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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: KAROL KOZIOL Email/Phone: (315)569-6968

Speaker's Name: BIRGIT SPEH

Talk Title: CONSTRUCTION OF SOME MODULAR SYMBOLS

Date: 8 / 15 / 14 Time: 3 : 30 am / pm (circle one)

List 6-12 key words for the talk: MODULAR SYMBOLS, AUTOMORPHIC FORMS

Please summarize the lecture in 5 or fewer sentences: IN THIS TALK, MODULAR SYMBOLS ARE DEFINED, AND SEVERAL EXAMPLES ARE DISCUSSED.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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Some modular symbols

MSRI, August 2014

Birgit Speh
Cornell University

Outline

1. Generalized Modular symbols: An introduction

2. First example : trivial representations = invariant forms (with T.N.Venkatarama)

3. Second example (joint with J.Rohlfs;): A residual representation of GL(4,$\mathbb{R}$)

4. Third example (joint with J. Rohlfs) ; Eisenstein cohomology

5. Work in progress with T.Kobayashi, G= O(n,1)

Generalized Modular Symbols: An Introduction

G semi simple Lie groups (connected)
K max compact subgroup
$X = G/K$ symmetric space
D dimension of X

Let $i : H \hookrightarrow G$ be a semisimple (reductive) subgroup
$K_H = K \cap H$ max compact
$Y = H/K_H$ symmetric space
d dimension of Y

Then

$$i : Y \hookrightarrow X$$

Γ discrete torsion free subgroup. $\Gamma_H = \Gamma \cap H$

For simplicity assume for now that

- $\Gamma \backslash X$ compact and orientable

- $\Gamma_H \backslash Y$ compact and orientable

and we have

$$\mathfrak{i} : \ \Gamma_H \backslash Y \to \Gamma \backslash X$$

Let $[\Gamma_H \backslash Y]$ fundamental class of $\Gamma_H \backslash Y$

Definition: The Homology class

$$i_*[(\Gamma_H \backslash Y)] \in H_d(\Gamma \backslash X)$$

is called a

generalized modular symbol.

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generalized modular symbol.

Problem: Determine properties of the modular symbol, for example determine if the modular symbol is nontrivial

Investigating

$$\mathfrak{i}_* : H_*(\Gamma_H\backslash Y, \mathbb{C}) \to H_*(\Gamma\backslash X, \mathbb{C})$$

is equivalent to

$$\mathfrak{i}^* : H^*_{deRham}(\Gamma\backslash X, \mathbb{C}) \to H^*_{deRham}(\Gamma_H\backslash X, \mathbb{C})$$

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This gives a connection with representation theory and automorphic Representations

Review of representation theory automorphic forms and deRham cohomology of $\Gamma\backslash X$

For details see book by Borel /Wallach " Continuous cohomology, Discrete subgroups and Representations of reductive Groups"

$\mathfrak{g}$, $\mathfrak{k}$ Lie algebras of G and K
$\mathfrak{g} = \mathfrak{k} \oplus p$ Cartan decomposition
$\mathcal{E}^p(\Gamma\backslash G/K)$ p-forms on $\Gamma\backslash X$

$$\begin{aligned}\mathcal{E}^p(\Gamma\backslash G/K) &= (C^\infty(\Gamma\backslash G) \otimes p^*)^K \\ &= \mathrm{Hom}_K(\bigwedge^* p, C^\infty(\Gamma\backslash G))\end{aligned}$$

On the other hand

$$\Gamma\backslash G \text{ compact} \qquad \Rightarrow \qquad L^2(\Gamma\backslash G) = \oplus m_\pi \pi$$

where π runs over all irreducible unitary representations of G and
$m_\pi = \dim \mathrm{Hom}_G(\pi, L^2(\Gamma\backslash G)) < \infty$.

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By Matsushima Murakami showed

$$H^*(\Gamma\backslash X, \mathbb{C}) = \oplus_\pi H^*(\mathfrak{g}, K, \pi)^{m_\pi}.$$

Here $H^*(\mathfrak{g}, K, \pi)$ is referred to as the $(\mathfrak{g}, K)$–cohomology of π.

Unitary representations with nontrivial $(\mathfrak{g}, K)$ −cohomology are classified and well understood by the work of Vogan-Zuckerman.

Examples:

- Trivial representation Id. In this case $H^*(\mathfrak{g}, K, Id)$ is represented by invariant forms on X.

- $G = Sl(2, \mathbb{R})$, discrete series representations corresponding to cusp form of weight 2 has with nontrivial in degree 1.

- $G = Sl(2, R)$, B upper triangular matrices The principal series representation $I(B)$ induced from the trivial representation of B has nontrivial cohomology in degree 0, 1.

We have a non degenerate pairing

$$H^*(\Gamma\backslash X, \mathbb{C}) \times H_*(\Gamma\backslash X, \mathbb{C}) \to \mathbb{C}$$

by integration.

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Remarks:

We may consider this as a pairing between automorphic forms (representations) $\pi_{\mathbf{A}}$ and cycles.

If everything in on the left hand side is defined over a number field K then we can deduce information about the right hand side. The integral is often related to a special value of a L-function.

(see the work of Shimura, Harder ....)

Modular symbols have been considered in the work of

G. Shimura
G. Harder
A. Borel
Manin-Drinfeld
Muzur-Swinnerton-Dyer
A. Ash +Borel
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but mostly for noncompact discrete subgroups Γ.

Remarks about not cocompact subgroups Γwith finite volume (for example most arithmetic groups)

In this case $L^2(\Gamma\backslash G) \neq \oplus m_\pi \pi$

but

$$L^2(\Gamma\backslash G) = L^2_{cusp}(\Gamma\backslash G) \oplus L^2_{res}(\Gamma\backslash G) \oplus L^2_{cont}(\Gamma\backslash G)$$

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$L^2_{cusp}(\Gamma\backslash G) = \oplus_{\pi_{\mathbf{A}} \in L^2_{cusp}(\Gamma\backslash G)} \pi_{\mathbf{A}}$ is a direct sum of cusp representations (forms)
$L^2_{res}(\Gamma\backslash G) \oplus_{\pi_{\mathbf{A}} \in L^2_{res}(\Gamma\backslash G)} \pi_{\mathbf{A}}$ is a direct sum of residual representations.

$$H^*(\Gamma\backslash X, \mathbb{C}) = H^*_{cusp}(\Gamma\backslash X, \mathbb{C}) \oplus H^*_{res}(\Gamma\backslash X, \mathbb{C}) \oplus H^*_{Eis}(\Gamma\backslash X, \mathbb{C})$$

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where

$$H^*_{cusp}(\Gamma\backslash X, \mathbb{C}) = \oplus_{\pi_{\mathbf{A}} \in L^2_{cusp}(\Gamma\backslash G)} H^*(\mathfrak{g}, K, \pi_{\mathbf{A}})$$

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BUT the map

$$\oplus_{\pi_{\mathbf{A}} \in L^2_{res}(\Gamma\backslash G)} H^*(\mathfrak{g}, K, \pi_{\mathbf{A}}) \to H^*_{res}(\Gamma\backslash X, \mathbb{C})$$

is NOT INJECTIVE.

First example : Generalized modular symbols related to invariant forms (joint work with Venkataramana)

$\mathbf{G}$, $\mathbf{H}$ semi simple algebraic groups defined over $\mathbb{Q}$,
$G = \mathbf{G}(\mathbb{R})$, $H = \mathbf{G}(\mathbb{R})$
Γ torsion free congruence subgroup.
Assume that G, H are connected.

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$$\Gamma\backslash G \text{ is not compact.}$$

$\pi = id$ the trivial representation. It is in $L^2_{res}(\Gamma\backslash G)$

Jens Franke determined the contribution of $H^*(\mathfrak{g}, K, Id)$ to $H^*(\Gamma\backslash X, \mathbb{C})$ i.e the contribution of the invariant forms to the deRham cohomology of $\Gamma\backslash G$

Compactly supported cohomology classes in $H_c^*(\Gamma\backslash X)$ maybe pulled back and integrated on $\Gamma_H\backslash Y$. Integration on $\Gamma_H\backslash Y$ is a linear form on $H_c^*(\Gamma\backslash X)$. By Poincare duality $[\Gamma_H\backslash Y] \in H^{D-d}(\Gamma\backslash X)$.

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We prove a nonvanishing criterion for modular symbols involving the compact dual of X.

In a special case we obtain:

Theorem 1. *Let E be a totally imaginary number field, $\mathbf{G} = R_{E/\mathbb{Q}}(Sl_{2n})$ and $\mathbf{H} = R_{E/\mathbb{Q}}(Sp_{2n})$. Then the modular symbol $[\Gamma_H\backslash Y]$ doesn't vanish for some congruence subgroup Γ.*

Theorem 2. *Let* $\mathbf{G}$ *be the split symplectic group* Sp_{2g} *over* $\mathbb{Q}$ *and let* $\mathbf{H} = \Pi_i Sp_{2g_i}$ *with* $\sum g_i = g$ *Then the modular symbol* $[\Gamma_H \backslash Y]$ *is non zero for a suitable congruence subgroup* Γ*.*

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The cohomology of the locally symmetric space is a module of the Hecke algebra and hence the Hecke algebra $\mathbf{G}(\mathbf{A}_f)$ also operates on the modular symbols.

Theorem 3. *Suppose that $G = U(1, q)$ and $H = U(1, r)$ with r=q-2 or r= q-1. Then there exists a congruence subgroup Γ such that the $\mathbf{G}(\mathbf{A}_f)$-translates of the modular symbol $[\Gamma_H \backslash Y]$ is infinite dimensional.*

Second example: Symplectic modular symbols for $Gl(4,\mathbb{R})$.

Joint with J. Rohlfs

$G = GL(4,\mathbb{R})$ (reductive and disconnected
H a symplectic group compatible with the choice of the maximal compact subgroup
Γ torsion free congruence subgroup.

There exists a unique infinite dimensional representations $\pi_{\mathbf{A}}$ in the residual spectrum with nontrivial $(\mathfrak{g}, K)$-cohomology in degree 3. Furthermore we have

$$0 \to H^3(\mathfrak{g}, K, \pi_{\mathbf{A}}) \to H^3(\Gamma\backslash X)$$

Let $[\omega_{\pi_{\mathbf{A}}}] \in H_c^6(\Gamma\backslash X)$ the class defined by its Poincare dual.

Theorem 4. *For Γ small enough $[\Gamma_H\backslash Y]$ is a nontrivial modular symbol and defines a linear functional on $H_c^6(\Gamma\backslash X)$.*

Conjecture: We conjecture that the value of the integral of $\omega_{\pi_{\mathbf{A}}}$ is related to special values of Rankin convolutions of certain cusp forms of Gl_2 and is closely related to some integrals of Jacquet/Rallis.

Third example: Eisenstein cohomology, pseudo Eisenstein cohomology and modular forms (joint with J. Rohlfs)

Roughly speaking:
The nontrivial classes in $H^*_{Eis}(\Gamma\backslash X, \mathbb{C})$ are represented by harmonic forms which have a non trivial restriction to a face of Borel -Serre compactification of $\Gamma\backslash X$. These forms are constructed similar to Eisenstein series and related to induced representations. (see work of Harder and Schwermer)

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The classes in the Poincare dual of $[H^*_{Eis}(\Gamma\backslash X, \mathbb{C})]$ are represented by compactly supported "pseudo Eisenstein forms". We denoted this subspace of the cohomology with compact support

$$[H^*_{pseudoEis}(\Gamma\backslash X, \mathbb{C})]$$

Using $H_c^*(\Gamma\backslash X, \mathbb{C}$ and $H^*_{pseudoEis}(\Gamma\backslash X, \mathbb{C})$ we prove as a special case a generalization of a theorem of Ash-Borel

Theorem 5. *The Modular symbol attached to the fundamental class of the Levi factor of a parabolic subgroup defined over $\mathbb{Q}$ doesn't vanish. It defines a non trivial linear form on*

$$[H^*_{pseudoEis}(\Gamma\backslash X, \mathbb{C})]$$

.

Work in progress with T.Kobayashi G=O(n,1),

For simplicity G=O(2m,1).
Remark: $G/G^0 = \mathbb{Z}_2 \times \mathbb{Z}_2$ and there are 4 inequivalent one dimensional representations.

Recall Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus p$. We have $p = \mathbb{R}^n$ and the representation of K on $\bigwedge^n p$ may not be trivial.

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Recall Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus p$. We have $p = \mathbb{R}^n$ and the representation of K on $\bigwedge^n p$ may not be trivial.

Vogan-Zuckerman results imply that there exist exactly m inequivalent representations U_i with non trivial $(\mathfrak{g}, K)$ cohomology.

We choose a numbering so that

$$H^j(\mathfrak{g}, K, U_i) = \mathbb{C} \qquad \text{if } j = i \text{ or } j = n - i$$

and zero otherwise.

We pick the Casselman-Wallach realization of representation on a Frechet space.

Theorem 6. *Let V_0 be one-dimensional representation of H=O(n-i) with nontrivial $(\mathfrak{h}, K_H)$-cohomology. Then*

$$\mathit{Hom}_H(U_i, V_0) \geq 1.$$

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Theorem 6. *Let V_0 be one-dimensional representation of H=O(n-i) with nontrivial $(\mathfrak{h}, K_H)$-cohomology. Then*

$$\mathit{Hom}_H(U_i, V_0) \geq 1.$$

Analyzing this H equivariant homomorphism we can show

Corollary 7. *We obtain a nontrivial map for the restriction of forms*

$$\bigwedge^{n-i} p \otimes_K U_i \to \bigwedge^{n-i} p \otimes_{K_H} V_0$$

Using this, the techniques developed in example 2 ,results about representations in the residual spectrum and pseudo Eisenstein series we expect to prove

Theorem 8. *? ? ? Suppose that $\mathbb{G} = O(2m, 1)$ and $\mathbb{H}$ be O(k,1) with $m < k < n$ Γ a torsion free arithmetic group.. For Γ small enough the module of $[\Gamma_H \backslash Y]$ under the Hecke algebra contains a nontrivial modular symbol.*

This is still work in progress, not all the details have been checked yet!!

A similar statement to Theorem 6 and 7 is also true for O(2m+1,1).