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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: KAROL KOZIOL **Email/Phone:** (315)569-6968

Speaker's Name: LING LONG

Talk Title: GENERALIZED LEGENDRE CURVES AND ABELIAN VARIETIES WITH QUATERNIONIC MULTIPLICATION

Date: 8 / 14 / 14 **Time:** 9 : 30 am / pm (circle one)

List 6-12 key words for the talk: LEGENDRE CURVES, JACOBIANS, QM, POINT COUNTING, GALOIS REPRESENTATIONS

Please summarize the lecture in 5 or fewer sentences: IN THIS TALK GENERALIZED LEGENDRE CURVES ARE DISCUSSED AND POINT COUNTS OVER FINITE FIELDS ARE OBTAINED. ADDITIONALLY, CONDITIONS ARE GIVEN FOR THE ~~[illegible]~~ ENDOMORPHISM ALGEBRA OF AN ASSOCIATED ABELIAN VARIETY TO BE A QUATERNION ALGEBRA.

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Generalized Legendre Curves and Abelian Varieties with Quaternionic Multiplication

Ling Long, joint with
Alyson Deines, Jenny Fuselier, Holly Swisher, Fang-Ting Tu
MSRI Connections for Women: New Geometric Methods in Number Theory and Automorphic Forms

Arithmetic triangle groups

- The *triangle group* (e_1, e_2, e_3) with $2 \leq e_1, e_2, e_3 \leq \infty$:

$$\langle x, y \mid x^{e_1} = y^{e_2} = (xy)^{e_3} = id \rangle.$$

- Such a Γ is called *arithmetic* if it has a unique embedding to $SL_2(\mathbb{R})$ with image either commensurable with $PSL_2(\mathbb{Z})$ or related to an order of a totally indefinite quaternion algebra over a totally real field. Arithmetic triangle groups Γ have been classified by Takeuchi. Γ acts on the upper half plane. The quotient space is a modular curve when at least one of e_i is ∞; otherwise, it is a Shimura curve.
- Shimura curve for Γ parametrizes isomorphism classes of 2-dimensional abelian varieties so that for each fiber the endomorphism ring contains the quaternion algebra associated with Γ.

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(∞, ∞, ∞) and its modular curve

The arithmetic triangle group (∞, ∞, ∞) is isomorphic to $\Gamma(2)$. A model of the modular curves for $\Gamma(2)$ is the Legendre family of curves

$$y^2 = x(1-x)(1-\lambda x).$$

A period for this curve is

$$p(\lambda) = \pi \sum_{k \geq 0} \binom{2k}{k}^2 \frac{\lambda^k}{16^k},$$

which is a hypergeometric series.

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Hypergeometric series

- $$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; \lambda\right] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{\lambda^k}{k!}, $$

  where $(a)_k = a(a+1)\cdots(a+k-1)$. We assume $a, b, c \in \mathbb{Q}$.
- It is a solution of

  $$ HDE(a, b, c; \lambda) : \lambda(1-\lambda)F'' + [(a+b+1)\lambda - c]F' + abF = 0, $$

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Schwarz's theorem

Theorem (Schwarz)

Let f, g be two independent solutions to $HDE(a, b; c; \lambda)$ at a point $z \in \mathfrak{H}$, and let $r_1 = |1-c|$, $r_2 = |c-a-b|$, and $r_3 = |a-b|$. Then the Schwarz map $D = f/g$ gives a bijection from $\mathfrak{H} \cup \mathbb{R}$ onto a curvilinear triangle with vertices $D(0), D(1), D(\infty)$, and corresponding angles $r_1\pi, r_2\pi, r_3\pi$.

When r_1, r_2, r_3 are rational numbers in the lowest form (with $0 = \frac{1}{\infty}$), let e_i be the denominators of r_1, r_2, r_3 arranged in the non-decreasing order, the monodromy group is isomorphic to the triangle group (e_1, e_2, e_3).

Example

When $a = \frac{1}{6}, b = \frac{1}{3}, c = \frac{5}{6}$, $r_1 = |1-c| = \frac{1}{6}$, $r_2 = |c-a-b| = \frac{1}{3}$, $r_3 = |a-b| = \frac{1}{6}$. The corresponding triangle group is $(3, 6, 6)$.

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Generalized Legendre Curves

- Euler's integral representation of the ${}_2F_1$ with $c > b > 0$

$$\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-\lambda x)^{-a}dx = {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} ; \lambda\right] B(b, c-b), \quad (1)$$

where $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the Beta function.

- Following Wolfart , this integral can be realized as a *period* of

$$C_\lambda^{[N;i,j,k]} : y^N = x^i(1-x)^j(1-\lambda x)^k,$$

where $N = lcd(a, b, c)$, $i = N(1-b)$, $j = N(1+b-c)$, $k = Na$.

- The point counting on this curve is very explicit.
- Example: associated to $a = \frac{1}{6}, b = \frac{1}{3}, c = \frac{5}{6}$ is $C_\lambda^{[6;4,3,1]}$.

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Petkoff-Shiga,'s result for (3,6,6)

By Petkoff-Shiga, for any $\lambda \in \overline{\mathbb{Q}}$, the Picard curve

$$C(\lambda):\ w^3 = (z^2 - 1/4)\left(z^2 - \lambda/4\right)$$

satisfies that

- the Jacobian $J(\lambda) = E'(\lambda) \oplus A'(\lambda)$
- $E'(\lambda):\ w^3 = (z - 1/4)\,(z - \lambda/4)$ is a CM elliptic curve
- for each $\lambda \in \overline{Q}$, $End_0(A'(\lambda)) = End(A(\lambda)) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains $\left(\frac{-3,2}{\mathbb{Q}}\right)$ the quaternion algebra associated with $(3, 6, 6)$.

A motivation

Question:

Given a hypergeometric differential equation $HDE(a, b, c; \lambda)$ whose monodromy group is an arithmetic triangle group $\Gamma = (e_1, e_2, e_3)$, does the Jacobian of the associated the generic generalized Legendre curve contains a 2-dimensional sub-abelian variety whose endomorphism algebra contains the quaternion algebra associated with Γ?

Example: $C_\lambda^{[6;4,3,1]}$ with $\Gamma = (3,6,6)$

For any $\lambda \in \mathbb{Q}$, the curve $C_\lambda^{[6;4,3,1]}: \ y^6 = x^4(1-x)^3(1-\lambda x)$, its Jacobian variety is decomposed as

$$\mathrm{Jac}(X_\lambda^{[6;4,3,1]}) = E(\lambda) \oplus A(\lambda),$$

where

$$E(\lambda): \ y^3 = x^4(1-x)^3(1-\lambda x)$$

is a CM elliptic curve.

Comparison with the Picard curve by Petkoff-Shiga

Local L-functions with $\lambda = 2$

p	$L_p(X_\lambda^{[6;4,3,1]}, T)$	$L_p(C(\lambda), T)$
7	$E:\ 7T^2 + 4T + 1$ $A:\ (7T^2 - 2T + 1)^2$	$E':\ 7T^2 + 4T + 1$ $A':\ (7T^2 - 2T + 1)^2$
11	$E:\ 11T^2 + 1$ $A:\ 121T^4 - 2T^2 + 1$	$E':\ 11T^2 + 1$ $A':\ 121T^4 - 2T^2 + 1$
13	$E:\ 13T^2 - 2T + 1$ $A:\ 169T^4 - 14T^2 + 1$	$E':\ 13T^2 - 2T + 1$ $A':\ 169T^4 - 14T^2 + 1$
17	$E:\ 17T^2 + 1$ $A:\ 289T^4 + 16T^2 + 1$	$E':\ 17T^2 + 1$ $A':\ 289T^4 + 16T^2 + 1$
19	$E:\ 19T^2 - 8T + 1$ $A:\ 361T^4 + 10T^2 + 1$	$E':\ 19T^2 - 8T + 1$ $A':\ 361T^4 + 10T^2 + 1$

Using counting points techique based on formal group laws, we can show that

Theorem (Deines, L., Fuselier, Swisher, Tu)

Let $\lambda \in \mathbb{Q}$, ℓ *be prime, and* ρ_ℓ, ρ'_ℓ *the 4-dimensional* ℓ*-adic Galois representations of* $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *arising from* $A(\lambda)$ *and* $A'(\lambda)$*, respectively. If both* ρ *and* ρ' *are absolutely irreducible, then they are isomorphic.*

Generalized Legendre Curves and Galois Representations.

Let $X(\lambda) = X_\lambda^{[N;i,j,k]}$ be the smooth model of $C_\lambda^{[N;i,j,k]}$. Its genus is

$$g = 1 + N - \frac{\gcd(N, i+j+k) + \gcd(N,i) + \gcd(N,j) + \gcd(N,k)}{2}. \quad (2)$$

Let $J_\lambda^{[N;i,j,k]}$ be the Jacobian of variety of $X(\lambda) = X_\lambda^{[N;i,j,k]}$.

Decomposition of the Jacobian variety

For any Nth root ζ, $A_\zeta : (x, y) \mapsto (x, \zeta^{-1} y)$ is an order N automorphism on $C_\lambda^{[N;i,j,k]}$.

For any $n \mid N$, $C_\lambda^{[N;i,j,k]}$ contains a quotient isomorphic to $C_\lambda^{[n;i,j,k]}$. Thus $J_\lambda^{[N;i,j,k]}$ contains a sub-abelian variety which is isomorphic to $J_\lambda^{[n;i,j,k]}$.

Let $J^{new}(\lambda)$ be the primitive part of $J_\lambda^{[N;i,j,k]}$ (over $\overline{\mathbb{Q}}$) so that its image in each $J_\lambda^{[n;i,j,k]}$ quotient is 0-dimensional. Archinard shows that the dimension of $J_\lambda^{[N;i,j,k]}$ is $\varphi(N)$, the Euler number of N.

Our goal is to study the Galois representations associated with $J^{new}(\lambda)$ and determine its endomorphism algebra.

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Our goal is to study the Galois representations associated with $J^{new}(\lambda)$ and determine its endomorphism algebra.

We will assume $\lambda \in \mathbb{Q}$ and first consider $\varphi(N) = 2$ cases so that $(\mathbb{Z}/N\mathbb{Z})^\times = \{1, N-1\}$. In this case, one can attach a compatible family of 4-dimensional Galois representation of $G_\mathbb{Q}$ associated with J_λ^{new}. When restricted to $G_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N))}$, it is isomorphic to $\sigma_1 \oplus \sigma_{N-1}$.

Our result for the case of $\varphi(N) = 2$

Theorem (Deines, L., Fuselier, Swisher, Tu)

Let $N = 3, 4, 6$, $1 \leq i, j, k < N$. Suppose $N \nmid i + j + k$. Then $J^{new}(\lambda)$ contains a quaternion algebra for all $\lambda \in \overline{\mathbb{Q}}$ (which can be determined explicitly) and if and only if, the quotient

$$B\left(\frac{N-i}{N}, \frac{N-j}{N}\right) \Big/ B\left(\frac{k}{N}, \frac{2N-i-j-k}{N}\right) \in \overline{\mathbb{Q}}.$$

Our method applies to $\varphi(N) > 2$ cases.

Decomposition of Galois representation

A_ζ also induces an action on the ℓ-adic Galois representation arising from the Tate module of $J(\lambda)$

$$\rho_\ell(\lambda) : G_{\mathbb{Q}} \to GL_{2g}(\overline{\mathbb{Q}}_\ell).$$

Consequently,

$$\rho_\ell(\lambda)|_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N))} = \bigoplus_{n=1}^{N-1} \sigma_n(\lambda)$$

where $\sigma_n(\lambda)$ is 2-dimensional when $(n, N) = 1$.

Galois representations via Gaussian hypergeo. series

Definition of Gaussian hypergeometric series by Greene

For characters A, B, and C in $\widehat{\mathbb{F}_q^\times}$ and $\lambda \in \mathbb{F}_q$, define

$$ {}_2F_1\begin{pmatrix} A & B \\ & C \end{pmatrix}; \lambda \Big)_q = \varepsilon(\lambda)\frac{BC(-1)}{q}\sum_{x\in\mathbb{F}_q} B(x)\overline{B}C(1-x)\overline{A}(1-\lambda x), $$

where

- ε is the trivial character, and
- we extend χ on $\mathbb{F}_q$ with $\chi(0) = 0$, for all $\chi \in \widehat{\mathbb{F}_q^\times}$.

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Proposition

If $A, B, C \in \widehat{\mathbb{F}_q^\times}$ $A, B \neq \varepsilon$, $A, B \neq C$, and $\lambda \in \mathbb{F}_q \setminus \{0, 1\}$,

$$J(A, \overline{A}C)\, {}_2F_1\begin{pmatrix} A & B \\ & C \end{pmatrix}; \lambda\Big)_q = AB(-1)\overline{C}(-\lambda)C\overline{AB}(1-\lambda)J(B, \overline{B}C)\, {}_2F_1\begin{pmatrix} \overline{A} & \overline{B} \\ & \overline{C} \end{pmatrix}; \lambda\Big)_q .$$

Counting points on generalized Legendre curves

Theorem

Let $p > 3$ be prime and $q = p^s \equiv 1 \pmod N$, and let i, j, k be natural numbers with $1 \le i, j, k < N$. Further, let $\xi \in \widehat{\mathbb{F}_q^\times}$ be a character of order N. Then for $\lambda \in \mathbb{F}_q \setminus \{0, 1\}$,

$$\#X_\lambda^{[N;i,j,k]}(\mathbb{F}_q) = 1 + q + q \sum_{m=1}^{N-1} \xi^{mj}(-1)\, {}_2F_1\left(\begin{matrix}\xi^{-km} & \xi^{im} \\ & \xi^{m(i+j)}\end{matrix}; \lambda\right)_q + n_0 + n_1 + n_{\frac{1}{\lambda}} + n_\infty - 4, \quad (3)$$

where $n_0, n_1, n_{\frac{1}{\lambda}}, n_\infty$ are the numbers of points on $X_\lambda^{[N;i,j,k]}$ from resolving the singularities $0, 1, \frac{1}{\lambda}, \infty$ respectively of $C_\lambda^{[N;i,j,k]}$

Theorem

Let N, i, j, k as before, $\lambda \in \mathbb{Q}$, p be any prime that is unramified for ρ_ℓ such that $\lambda \neq 0, 1 \pmod{\wp}$. Let $\wp$ be a prime of $\mathcal{O}_{\mathbb{Q}(\zeta_N)}$ above p and $q = |\mathcal{O}_{\mathbb{Q}(\zeta_N)}/\wp|$. Let $\xi \in \widehat{\mathbb{F}_q^\times}$ of order N and $Frob_\wp$ denotes the (arithmetic) Frobenius in $G_{\mathbb{Q}(\zeta_N)}$. For any n coprime to N, the values

$$\mathrm{Tr} Frob_\wp^{-1}(\sigma_n(\lambda)) \quad \text{and} \quad {}_2F_1\left(\begin{matrix}\xi^{-kn} & \xi^{in} \\ & \xi^{n(i+j)}\end{matrix}; \lambda\right)_q \cdot \xi^{nj}(-1)$$

agree up to different embeddings of $\mathbb{Q}(\zeta_N)$ in $\mathbb{C}$.

We will assume $\lambda \in \mathbb{Q}$ and first consider the $\varphi(N) = 2$ case so that $(\mathbb{Z}/N\mathbb{Z})^\times = \{1, N-1\}$. In this case, one can attach a compatible family of 4-dimensional Galois representations of $G_\mathbb{Q}$ associated with J_λ^{new}. When it is restricted to $G_{\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N)}$, it is isomorphic to $\sigma_1 \oplus \sigma_{N-1}$.

4-dimensional Galois representations with QM

When $End_0(J_\lambda^{new})$ is a quaternion algebra, then there are two semi-linear operators I, J acting on the 4-dimensional representation space of $End_0(J_\lambda^{new})$ such that I^2 and J^2 are scalars and $IJ = -JI$. In this case, we say the Galois representation admits QM.

Proposition

Assume that ρ_ℓ is a compatible family of 4-dimensional Galois representations of $G_\mathbb{Q}$ which admits QM. Let K be a number field such that both I, J are defined. Then $\rho_\ell|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$ is a direct sum of two isomorphic sub-representations.

Examples of 4-dimensional Galois representations with QM arising from noncongruence modular forms have been studied by A.O.L. Atkin, Wen-Ching Winnie Li, L. Tong Liu and Zifeng Yang.

Combing with that $\sigma_n(\lambda)$ can be computed using Gaussian $2F1$

Proposition

If $A, B, C \in \widehat{\mathbb{F}_q^\times}$ $A, B \neq \varepsilon$, $A, B \neq C$, and $\lambda \in \mathbb{F}_q \setminus \{0, 1\}$,

$$J(A, \overline{A}C)\, {}_2F_1\begin{pmatrix} A & B \\ & C \end{pmatrix}; \lambda\Big)_q = \\ AB(-1)\overline{C}(-\lambda)C\overline{AB}(1-\lambda)J(B, \overline{B}C)\, {}_2F_1\begin{pmatrix} \overline{A} & \overline{B} \\ & \overline{C} \end{pmatrix}; \lambda\Big)_q.$$

As $A = \xi^{-k}, B = \xi^i, C = \zeta^{(i+j)}$ for σ_1, one can conclude that if $\varphi(N) = 2$ and $End_0(J_\lambda^{new})$ is a quaternion algebra, then σ_1 and σ_{N-1} are differed by a character of $G_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N)}$ and consequently for each good prime $p \equiv 1 \mod N$

$$J(\xi^{in}, \xi^{jn})/J(\xi^{-kn}, \xi^{n(i+j+k)})$$

has to be a character in $\widehat{\mathbb{F}_p^\times}$.

Results on Gauss sums $g(\xi)$ and Jacobi sums

$$g(\chi)\overline{g(\chi)} = p, \quad \chi \neq \varepsilon$$

Hasse-Davenport relation: for $\ell \mid M$

$$g(\chi^{\ell a}) = (-1)^\ell \chi(\ell^{\ell a - M/2})\chi(2^{N/2})^{1-\ell} g(\chi^{M/2})^{1-\ell} \prod_{j=0}^{\ell-1} g(\chi^{a+(M/\ell)j})$$

Theorem (Yamamoto)

When $M \geq 4$ is an even number, and p is a prime such that M divides $p-1$, then the above two identities are the only two relations connecting the Gauss sums $g(\chi)$ for $\chi \in \widehat{\mathbb{F}_p^\times}$ satisfying $\chi^M = \varepsilon$, when considered as ideals.

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Jacobi sums and Beta functions

$$g(\chi)\overline{g(\chi)} = p,$$
$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(z\pi)}.$$

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Jacobi sums and Beta functions

If $z = \frac{i}{M}$ is a rational number, $\chi \in \widehat{\mathbb{F}_p^\times}$ of order M, we have the following dictionary

$$\begin{array}{ccc} \frac{i}{M} & \Longleftrightarrow & \chi^i \\ \frac{1}{2} & \Longleftrightarrow & \chi^{M/2} \\ \Gamma(\frac{i}{M}) & \Longleftrightarrow & g(\chi^i) \\ B(\frac{i}{M}, \frac{j}{M}) & \Longleftrightarrow & J(\chi^i, \chi^j). \end{array}$$

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Proposition. Let $M \geq 4$ be an even integer and M divides $p-1$ and let $\eta \in \widehat{\mathbb{F}_p^\times}$ of order M. Let $A = \eta^i, B = \eta^j, C = \eta^k$ be characters such that none of $A, B, C, \overline{A}C, \overline{B}C$ are trivial. If $J(\eta^j, \eta^{k-j})/J(\eta^i, \eta^{k-i})$ is a character for each prime p with $p \equiv 1 \mod M$, then $B(\frac{j}{M}, \frac{k-j}{M})/B(\frac{i}{M}, \frac{k-i}{M})$ is an algebraic number.

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In conclusion, when $\varphi(N) = 2$ and $End_0(J_\lambda^{new})$ contains a quaternion algebra then for each good prime $p \equiv 1 \mod N$ and ξ an order N character in $\widehat{\mathbb{F}_p^\times}$

$$J(\xi^{in}, \xi^{jn})/J(\xi^{-kn}, \xi^{n(i+j+k)})$$

has to be a character and $B(\frac{N-i}{N}, \frac{N-j}{N})/B(\frac{N-k}{N}, \frac{2N-i-j-k}{N})$ has to be algebraic.

Conversely, by computing the periods of J_λ^{new} explicitly in terms of hypergeometric series. The following are 4 linearly independent periods of second kind on $J_1 \oplus J_{N-1}$

$$\tau_1 = B\left(\frac{N-i}{N}, \frac{N-j}{N}\right) {}_2F_1\left[\begin{matrix} \frac{k}{N} & \frac{N-i}{N} \\ & \frac{2N-i-j}{N} \end{matrix}; \lambda\right],$$

$$\tau_2 = (-1)^{\frac{k+j}{N}} \lambda^{\frac{i+j-N}{N}} B\left(\frac{i+j+k-N}{N}, \frac{N-k}{N}\right) {}_2F_1\left[\begin{matrix} \frac{j}{N} & \frac{i+j+k-N}{N} \\ & \frac{i+j}{N} \end{matrix}; \lambda\right]$$

$$\tau_3 = B\left(\frac{i}{N}, \frac{j}{N}\right) {}_2F_1\left[\begin{matrix} \frac{N-k}{N} & \frac{i}{N} \\ & \frac{i+j}{N} \end{matrix}; \lambda\right],$$

$$\tau_4 = (-1)^{\frac{2N-k-j}{N}} \lambda^{\frac{N-i-j}{N}} B\left(\frac{2N-i-j-k}{N}, \frac{k}{N}\right) {}_2F_1\left[\begin{matrix} \frac{N-j}{N} & \frac{2N-i-j-k}{N} \\ & \frac{2N-i-j}{N} \end{matrix}; \lambda\right],$$

Using Euler transformation for hypergeometric series,

$$\tau_4/\tau_1 = \alpha(\lambda)\frac{\Gamma\left(2 - \frac{i+j+k}{N}\right)\Gamma\left(\frac{k}{N}\right)}{\Gamma\left(1 - \frac{i}{N}\right)\Gamma\left(1 - \frac{j}{N}\right)}$$

and

$$\tau_2/\tau_3 = \alpha(\lambda)^{-1}\frac{\Gamma\left(\frac{i+j+k}{N} - 1\right)\Gamma\left(1 - \frac{k}{N}\right)}{\Gamma\left(\frac{i}{N}\right)\Gamma\left(\frac{j}{N}\right)},$$

where $\alpha(\lambda) = (-1)^{\frac{k+j}{N}}\lambda^{\frac{N-i-j}{N}}(1-\lambda)^{\frac{k+j-N}{N}}$.

Wüstholz's Theorem

- Let A be an abelian variety isogenous over $\overline{\mathbb{Q}}$ to the direct product $A_1^{n_1} \times \cdots A_k^{n_k}$ of simple, pairwise non-isogenous abelian varieties A_μ defined over $\overline{\mathbb{Q}}$, $\mu = 1, \ldots, k$.
- Let $\Lambda_{\overline{\mathbb{Q}}}(A)$ denote the space of all periods of differentials, defined over $\overline{\mathbb{Q}}$, of the first kind and the second on A.
- Then the vector space $\widehat{V}_A$ over $\overline{\mathbb{Q}}$ generated by 1, $2\pi i$, and $\Lambda_{\overline{\mathbb{Q}}}(A)$, has dimension

$$\dim_{\overline{\mathbb{Q}}} \widehat{V}_A = 2 + 4 \sum_{\nu=1}^{k} \frac{\dim A_\nu^2}{\dim_{\mathbb{Q}}(End_0 A_\nu)}.$$

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$$\dim_{\overline{\mathbb{Q}}} \widehat{V}_A = 2 + 4 \sum_{\nu=1}^{k} \frac{\dim A_\nu^2}{\dim_{\mathbb{Q}}(End_0 A_\nu)}.$$

Thus, if $B(\frac{N-i}{N}, \frac{N-j}{N})/B(\frac{N-k}{N}, \frac{2N-i-j-k}{N})$ is algebraic, then $\widehat{V}_A$ over $\overline{\mathbb{Q}}$ is at most 8 dimensional. Thus J_λ^{new} is either

- simple whose endomorphism algebra is at least 4-dimensional
- it is a direct summand of 2 isogenous 1-dimensional abelian varieties

Consequently, $End_0(J_\lambda^{new})$ is either

- a division algebra that contains a quaternion algebra
- a matrix algebra

The period matrix can determine whether the endomorphism algebra is a division algebra. For instance, we can determine that the endomorphism algebra for the primitive part of $J_\lambda^{[6;4,3,1]}$ is indeed $\left(\frac{-3,2}{\mathbb{Q}}\right)$.

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For a generic genus-2 curve $C_\lambda^{[3;1,2,1]}$, its endomorphism algebra is $M_2(\mathbb{Q})$ and one its period is $\pi \cdot {}_2F_1\left[\begin{matrix}\frac{1}{3}, \frac{2}{3} \\ 1\end{matrix}; \lambda\right]$ whose corresponding monodromy group $(3, \infty, \infty)$. Using Galois representation, we can show that

Theorem

Let $\lambda \in \mathbb{Q} \setminus \{0, 1\}$ and ρ be the 4-dimensional Galois representation of $G_\mathbb{Q}$ arising from the genus-2 curve $y^3 = x(x-1)^2(1-\lambda x)$. Let ρ' be the Galois representation of $G_\mathbb{Q}$ arising from the elliptic curve $y^2 + xy + \frac{\lambda}{27} = x^3$. Then ρ is isomorphic to $\rho' \oplus (\rho' \otimes \chi_{-3})$ where χ_{-3} is the quadratic character of $G_\mathbb{Q}$ with kernel $G_{\mathbb{Q}(\sqrt{-3})}$.

$X_\lambda^{[5;1,4,1]}$ and Hilbert modular forms

From computing the corresponding Galois representation, one can predict that its L-function is related to two Hilbert modular forms, which differ by embeddings of $\mathbb{Q}(\sqrt{5})$ to $\mathbb{C}$. From numeric data, we identified two Hilbert modular forms, which are labeled by Hilbert Cusp Form 2.2.5.1-500.1-a in the LMFDB online database.

p	$L_p(X(\lambda), T)$ over $\mathbb{Q}(\sqrt{5})$	Hecke eigenvalues
7	$(49T^4 + 10T^2 + 1)(49T^4 - 10T^2 + 1)$	-10
11	$(11T^2 - 2T + 1)^4$	$2, 2$
13	$(169T^4 + 1)^2$	0
17	$(289T^4 - 20T^2 + 1)(289T^4 + 20T^2 + 1)$	20
19	$\left(19T^2 - 5\left(\frac{1+\sqrt{5}}{2}\right)T + 1\right)\left(19T^2 - 5\left(\frac{1-\sqrt{5}}{2}\right)T + 1\right)$ $\left(19T^2 + 5\left(\frac{1+\sqrt{5}}{2}\right)T + 1\right)\left(19T^2 + 5\left(\frac{1-\sqrt{5}}{2}\right)T + 1\right)$	$5\left(\frac{1\pm\sqrt{5}}{2}\right)$
31	$\left(\left(31T^2 + \left(\frac{1+5\sqrt{5}}{2}\right)T + 1\right)\left(31T^2 + \left(\frac{1-5\sqrt{5}}{2}\right)T + 1\right)\right)^2$	$\frac{-1\pm5\sqrt{5}}{2}$
41	$\left(\left(41T^2 + \left(\frac{1+5\sqrt{5}}{2}\right)T + 1\right)\left(41T^2 + \left(\frac{1-5\sqrt{5}}{2}\right)T + 1\right)\right)^2$	$\frac{-1\pm5\sqrt{5}}{2}$

Thank you!