Title: An Example-Based Introduction to Shimura Varieties and Their Compactifications

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Modular Curves

$\mathfrak{h} = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$

G

$\Gamma \subset SL_2(\mathbb{Z})$
Congruence

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \gamma z = \dfrac{az+b}{cz+d}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} \sim \begin{pmatrix} \gamma z \\ 1 \end{pmatrix}$$

$$\mathfrak{h} \hookrightarrow \mathbb{P}^1(\mathbb{C})$$

$$\Gamma \backslash \mathfrak{h} \hookrightarrow \underbrace{\Gamma \backslash (\mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q}))}$$

compact Riemann surface = alg curve / $\mathbb{C}$

$\{\text{modular curves}\}_{\text{varying } \Gamma}$

Hecke symmetry

"Galois symmetry" (these curves are in fact / some # field)

More generally
("double coset spaces")

setup: G = "reductive alg. group" / $\mathbb{Q}$
s.t. we can talk about $G(\mathbb{A})$

$$\mathbb{A} = \underbrace{\mathbb{A}_\infty}_{\mathbb{R}} \times \underbrace{\mathbb{A}^\infty}_{\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}} \quad \leftarrow \text{away from } \infty$$

$$\hat{\mathbb{Z}} = \varprojlim_N (\mathbb{Z}/N\mathbb{Z})$$

For simplicity, we'll define G as a group functor over $\mathbb{Z}$.
(assign $G(R)$ to R, functorially).

any comm. ring

e.g. $GL_n(R)$ = invertible elt's in $\mathrm{End}_R(R^{\oplus n})$

$\cup$

n integer $SL_n(R)$ elt's of det 1

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

$2n \times 2n$

$$Sp_{2n}(R) = \{ g \in GL_{2n}(R) : {}^t g J_n g = J_n \}$$

$$\langle x, y \rangle = {}^t x J_n y$$

p, q integers

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \qquad O_{p,q}(R) = \{ g \in GL_{p+q}(R) : {}^t g I_{p,q} g = I_{p,q} \}$$

$\supset SO_{p,q}(R)$ det $= 1$

Suppose $G(\mathbb{R})$ acts on some D

Consider, for each open cpt. subgroup $\mathcal{U} \subset G(\mathbb{A}^\infty)$

$$X_{\mathcal{U}} := \underbrace{G(\mathbb{Q})}_{\text{acts diagonally}} \backslash (D \times G(\mathbb{A}^\infty)) / \underbrace{\mathcal{U}}_{\text{acts on } G(\mathbb{A}^\infty)}$$

Now consider $\{ X_{\mathcal{U}} \}_{\mathcal{U}}$ varying

$G(\mathbb{A}^\infty)$ action permutes the collection of $\mathcal{U}$'s

$\boxed{\text{with reasonable } D}$ $\quad \mathcal{U} \mapsto g \mathcal{U} g^{-1}$

So, $\varinjlim_{\mathcal{U}} H^*(X_{\mathcal{U}}) \circlearrowleft G(\mathbb{A}^\infty)$ "Hecke Symm."

What D's should we consider?

e.g. $SL_n(\mathbb{R}) \circlearrowright \mathcal{S}_n = \{ n \times n$ real matrices, pos. def. (symm.) $\det = 1 \}$

$g \in SL_n(\mathbb{R})$, $\quad X \longmapsto {}^t g X g$

action transitive

Stab. of $1_n = SO_n(\mathbb{R}) \Rightarrow \mathcal{S}_n = SL_n(\mathbb{R})/SO_n(\mathbb{R})$

For $n=2$, get $SL_2(\mathbb{R})/SO_2(\mathbb{R}) \cong \mathfrak{h}$

For gen. n, might get odd dim. space$/\mathbb{R}$ (e.g. $n=3$)

We'll give "nice" D s.t. quotients of D by some Γ is "algebraic".

Fact $\quad \#\left(G(\mathbb{Q})\backslash G(\mathbb{A}^\infty)/\mathcal{U}\right) < \infty$

$$\Rightarrow X_{\mathcal{U}} = G(\mathbb{Q})\backslash(D \times G(\mathbb{A}^\infty))/\mathcal{U}$$

$$= G(\mathbb{Q})\backslash(D \times (\textstyle\coprod_i G(\mathbb{Q}) g_i \mathcal{U}))/\mathcal{U}$$

$$\underset{\text{(exercise)}}{=} \coprod_i \Gamma_i \backslash D, \qquad \Gamma_i = G(\mathbb{Q}) \cap g_i \mathcal{U} g_i^{-1}$$

Want $\Gamma_i \backslash D$ to be <u>algebraic var</u>

($\Rightarrow$ Galois symm & nontrivial congruences)

We'll want D to be (finite unions of) Hermitian symm. domains

<u>Remark</u>: Will focus first on "connected components".

What D's should we consider?

<u>e.g.</u> ("Siegel case") $\quad G = Sp_{2n}(\mathbb{R}) \curvearrowright \mathfrak{h}_n$

$$\mathfrak{h}_n = \{ Z \in Sym_n(\mathbb{C}) : Im(Z) > 0 \}$$

Siegel upper-half space

$g \in Sp_{2n}(\mathbb{R})$, $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ $\qquad gZ = (AZ+B)(CZ+D)^{-1}$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} Z \\ 1 \end{pmatrix} = \begin{pmatrix} AZ+B \\ CZ+D \end{pmatrix} \sim \begin{pmatrix} gZ \\ 1 \end{pmatrix}$$

div by $(CZ+D)$

$$\mathfrak{h}_n = \left\{ Z \in \mathrm{Mat}_n(\mathbb{C}) : {}^t\!\begin{pmatrix} Z \\ 1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}}_{J_n} \begin{pmatrix} Z \\ 1 \end{pmatrix} = 0 \iff {}^tZ = Z \right.$$
$$\left. {}^t\!\begin{pmatrix} \bar{Z} \\ 1 \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} Z \\ 1 \end{pmatrix} < 0 \iff \mathrm{Im}(Z) > 0 \right\}$$

preserved by $Sp_{2n}(\mathbb{R})$

$Sp_{2n}(\mathbb{R}) \curvearrowright \mathfrak{h}_n$

to show transitivity: $\quad Z = X + iY$
(move all pts to iI_n)

can choose A s.t. this is iI_n

$$\begin{pmatrix} 1 & -X \\ 0 & 1 \end{pmatrix}\begin{pmatrix} Z \\ 1 \end{pmatrix} \sim \begin{pmatrix} iY \\ 1 \end{pmatrix} \qquad \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}\begin{pmatrix} iY \\ 1 \end{pmatrix} \sim \begin{pmatrix} iAY{}^tA \\ 1 \end{pmatrix}$$

Stab. at $iI_n = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in Sp_{2n}(\mathbb{R}) \right\}$

$$\cong U_n(\mathbb{R}) = \{ g \in GL_n(\mathbb{C}) : {}^t\bar{g}\, I_n\, g = I_n \}$$

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \longleftrightarrow A + iB$$

$$\Rightarrow \mathfrak{h}_n \cong Sp_{2n}(\mathbb{R}) / \mathcal{U}_n(\mathbb{R})$$

$$\mathcal{U}_n = \text{Cent. of } h_0 : \mathcal{U}_1(\mathbb{R}) \longrightarrow Sp_{2n}(\mathbb{R})$$
$$a + bi \longmapsto \begin{pmatrix} aI_n & -bI_n \\ bI_n & aI_n \end{pmatrix}$$

$\Downarrow$

$\mathfrak{h}_n \cong G(\mathbb{R}) \cdot h_0 \quad$ (Stab. under conj. action of $Sp_{2n}(\mathbb{R})$ on h_0)

$J = h_0(i)$ defines a "complex str" on $\mathbb{R}^{2n}$

Consider $\displaystyle A_{h_0} = \underbrace{(\mathbb{R}^{\oplus 2n})}_{\mathbb{C}\text{-v.sp.}} / \underbrace{\mathbb{Z}^{\oplus 2n}}_{\text{lattice}}$

(for $z = iI_n$) $\qquad \cong \mathbb{C}^{\oplus n} / (\mathbb{Z}^{\oplus n} z + \mathbb{Z}^{\oplus n})$

If $h = g h_0$, then for $z = g \cdot (iI_n)$

$$A_h = \mathbb{R}^{\oplus 2n} / \mathbb{Z}^{\oplus 2n}$$

complex str. $J = h(i)$

"polarized"

$$\cong \mathbb{C}^{\oplus n} / (\mathbb{Z}^{\oplus n} z \oplus \mathbb{Z}^{\oplus n})$$

<u>Fact</u>: $(A_h, \text{polar.}) \cong (A_{h'}, \text{polar.})$

$\iff h = \gamma h'$ for some

$\gamma = Sp_{2n}(\mathbb{Z}$

$\Rightarrow$ points of $\underbrace{\Gamma}_{\substack{\shortparallel \\ Sp_{2n}(\mathbb{Z})}} \backslash \mathfrak{h}_n$ param. (principal) polarized ab. var / $\mathbb{C}$

$\boxed{\text{$\Gamma$ congr. subgroup def'd by cond. mod n}}$

$\Rightarrow$ cond. on "n-torsion pt's" on ab. var.'s ("level str.'s)

<u>tomorrow</u>

1. "PEL type"?
2. "Hodge/ab. type"?
3. "exceptional type"?