Title: An Example-Based Introduction to Shimura Varieties and Their Compactifications

Speaker:

Date: 2014.08.19   Time: 11:00 am

last time:

$$Sp_{2n}(\mathbb{R}) \curvearrowright \mathfrak{h}_n = \{ Z \in Sym_n(\mathbb{C}) \text{ s.t. } Im(Z) > 0 \}$$

$$GSp_{2n}(\mathbb{R}) \curvearrowright \mathfrak{h}^{\pm} = \left\{ Z \in Sym_n(\mathbb{C}) \text{ s.t. } \begin{matrix} Im(Z) > 0 \\ Im(Z) < 0 \end{matrix} \right\}$$

$$GSp_{2n}(\mathbb{R}) = \left\{ \begin{matrix} (g, r) \in GL_{2n}(\mathbb{R}) \times \mathbb{R}^{\times} \\ {}^t g J_n g = r J_n \end{matrix} \right\}$$

r = "similitude"

$$h_0 : U_1(\mathbb{R}) \longrightarrow Sp_{2n}(\mathbb{R})$$
$$a+bi \longmapsto \begin{pmatrix} aI_n & -bI_n \\ bI_n & aI_n \end{pmatrix}$$

better for VHS

$$h_0 : \mathbb{C}^{\times} \longrightarrow GSp_{2n}(\mathbb{R})$$
$$a+bi \longmapsto \text{same}$$

VHS: variation of Hodge structures

EX 2 $\qquad I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$

$$U_{p,q}(\mathbb{R}) = \{ g \in GL_{p+q}(\mathbb{C}) : {}^t\bar{g}\, I_{p,q}\, g = I_{p,q} \}$$

$(p \geq q)$

$$\mathcal{D}_{p,q} = \left\{ u \in M_{p,q}(\mathbb{C}) : {}^t\begin{pmatrix} \bar{u} \\ 1 \end{pmatrix} \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \begin{pmatrix} u \\ 1 \end{pmatrix} = {}^t\bar{u}u - I_q < 0 \right\}$$

(bounded realization)

e.g. $p = q = 1$, $\quad \mathcal{D}_{1,1} = \{ |u| < 1 \} \subset \mathbb{C}$

alternative realization

$$U_{p,q}(\mathbb{R}) = \{ g \in GL_{p+q}(\mathbb{C}) : {}^t\bar{g}\, J_{p,q}\, g = J_{p,q} \}$$

different det

size $p+q$ $\to J_{p,q}$ $\begin{pmatrix} & & I_q \\ & S & \\ -I_q & & \end{pmatrix}$ $\quad \underbrace{-iS}_{\text{Hermitian}} > 0 \leftarrow$ size $p-q$

Skew-Herm

$$U_{p,q}(\mathbb{R}) \curvearrowright \mathfrak{h}_{p,q} = \left\{ \begin{array}{c|c} & \\ q & \boxed{Z} \\ p-q & \boxed{W} \end{array} \in M_{q\times q}(\mathbb{C}) \times M_{(p-q)\times q}(\mathbb{C}) \right\}$$

$$-i\, {}^t\begin{pmatrix} \bar{Z} \\ \bar{W} \\ 1 \end{pmatrix} J_{p,q} \begin{pmatrix} Z \\ W \\ 1 \end{pmatrix} = -i\,({}^t\bar{Z} - Z + {}^t\bar{W} S W) < 0$$

Rem: pairing: Q on complexified v. sp. $Q(v,v) = 0 - i\,Q(\bar{v}, v) < 0$

assume $p = q$

$$\mathfrak{h}_{p,q} = \{ Z \in M_{q,q}\mathbb{C} : -i({}^t\bar{Z} - Z) < 0 \Leftrightarrow \text{"}\mathrm{Im}(Z) > 0\text{"} \}$$

"Skew-Herm. part."

$$M_{q,q}(\mathbb{C}) \cong \mathrm{Herm}_n(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$$

Ex. 3 $\quad SO^*_{2n}(\mathbb{R})$

$\quad\rotatebox{90}{$\subset$}$ = the subgroup of $SO_{2n}(\mathbb{C})$

preserving the Skew-Herm. pairing def'd by $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

$$\mathfrak{h}_{SO^*_{2n}} = \left\{ Z \in M_n(\mathbb{C}) : {}^t\begin{pmatrix} Z \\ 1 \end{pmatrix}\begin{pmatrix} Z \\ 1 \end{pmatrix} = {}^tZZ + 1 = 0,\ -i\,{}^t\begin{pmatrix} \bar{Z} \\ 1 \end{pmatrix}\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\begin{pmatrix} Z \\ 1 \end{pmatrix} = -i({}^t\bar{Z} - Z) < 0 \right\}$$

When $n = 2k$, then $SO^*_{4k}(\mathbb{R})$

can be realized as the subgroup

$Sp_{2k}(\mathbb{H})$ of $2k \times 2k$ matrices over $\mathbb{H}$

Hamiltonians

preserving the Skew-Herm. pairing def by $\begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$

Then $\mathfrak{h}_{SO^*_{4k}} \cong \left\{ \begin{matrix} Z \in \mathrm{Herm}_k(\mathbb{H}) \\ \mathrm{Im}(Z) > 0 \end{matrix} \right\}$

Ex. 4 $\quad SO_{p,q}(\mathbb{R}) = \left\{ \begin{matrix} g \in SL_{p+q}(\mathbb{R}) : \\ {}^tg I_{p,q} g = I_{p,q} \end{matrix} \right\} \qquad I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$

$\quad\rotatebox{90}{$\subset$}$

$V = \mathbb{R}^{p+q}$

$$\mathfrak{h}_{SO_{p,q}} = \left\{ v \in V_{\mathbb{C}} : \begin{matrix} {}^tv I_{p,q} v = 0 \\ {}^t\bar{v} I_{p,q} v < 0 \end{matrix} \right\} \Big/ \mathbb{C}^\times$$

$V = \mathbb{R}^{p+q}$

(Working over $\mathbb{P}^1(V_{\mathbb{C}})$)

(Hermitian if $q = 2$)

Decompose $\quad V = \mathbb{R}^{\oplus p} \oplus \mathbb{R}^{\oplus 2}$

with pairings $\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ $\quad \begin{pmatrix} I_{p-1} & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$

($q=2$ for now) $\qquad$ (signature $(1,1)$)

Denote coord.'s of $V_{\mathbb{C}}$ by $z_1, \dots, z_p, w_1, w_2$

Then $$\mathfrak{h}_{so_{p,q}} = \left\{ \begin{array}{l} z_1^2 + \cdots + z_{p-1}^2 - z_p^2 + w_1 w_2 = 0 \\ |z_1|^2 + \cdots + |z_{p-1}|^2 - |z_p|^2 + \bar{w}_1 w_2 + w_1 \bar{w}_2 < 0 \end{array} \right\}$$

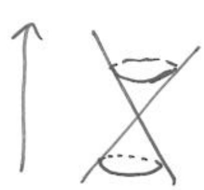
If w_2 were 0, then

$$|z_1|^2 + \dots + |z_{p-1}|^2 < |z_p|^2 = |z_1^2 + \dots + z_{p-1}^2| \text{ can't be true.}$$

May assume $w_2 = 1$ (by div. nonzero values)

$$\Rightarrow w_1 = z_p^2 - z_1^2 - \dots - z_{p-1}^2$$

$$\Rightarrow \underset{(q=2)}{\mathfrak{h}_{so_{p,q}}} \cong \left\{ \begin{array}{l} z \in \mathbb{C}^{\oplus p} : y = \mathrm{Im}(z) > 0 \\ \text{in the sense that, if } y_i = \mathrm{Im}(z_i) \\ \quad y = (y_1, \dots, y_p), \text{ then } y_p^2 > y_1^2 + \dots + y_{p-1}^2 \end{array} \right\}$$



y_p

light-cone
(two connected components)
the subgroup of $SO_{p,q}(\mathbb{R})$ of elt's with spinor norm $+1$ preserves the $+$ component

$Spin_{p,q}$ can be embedded in a (very large) symplectic group

$\uparrow$

"Hodge-type"

(parametrized by ab. var.'s w/ Hodge tensors)

($SO_{p,q}$ can <u>not</u>)

(abelian type: Same conn. comp. as Hodge-type <u>not</u> the same rat'l str.)

Sp_{2n} --- Type C

$U_{p,q}$ Type A
(over $\mathbb{C}$, $\cong GL_{p+q}$)

SO^*_{2n} Type D (type $D_{\mathbb{H}}$)

$SO_{p,2}$ Type B p odd
or D p even (type $D_{\mathbb{R}}$)

(<u>Rem</u>: If have mixture of types $D_{\mathbb{R}}$ and $D_{\mathbb{H}}$ in the same $\mathbb{Q}$-simple factor of the group
<u>Not</u> abelian type)

Two more $\mathbb{R}$-simple cases (of exceptional type) E_6, E_7.

$$\underset{\substack{\uparrow \\ \text{Cartan index}}}{E_{7(-25)}} \circlearrowright \mathfrak{h}_{E_7} = \left\{ \begin{array}{c} Z \in Herm_3(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C} \\ Im(Z) > 0 \end{array} \right\}$$

27-dim'l$/\mathbb{C}$

$$E_{6(-14)} \circlearrowright \mathfrak{h}_{E_6} \leftarrow \dim/_{\mathbb{C}} = 16$$

We have compactifications of quotients like $\Gamma\backslash D$

$$
\begin{tikzcd}
X_\Gamma = \Gamma\backslash D \arrow[r, hook] & (\Gamma\backslash D)^{tor}_\Sigma \arrow[dr] & \arrow[l, dashed] \times \arrow[r, dashed] & (\Gamma\backslash D)^{BS} \arrow[dl] \\
& & (\Gamma\backslash D)^{min} &
\end{tikzcd}
$$

can be sm/proj. for good Σ

Borel-Serre

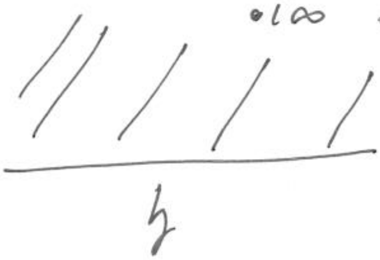
manifolds w/ "corners"

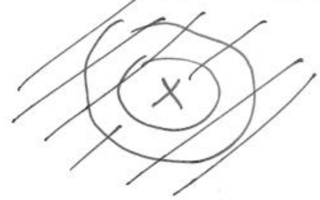
$\boxed{\text{Sh. var's}}$ are (q-proj.) var's

Satake-Baily-Borel normal $\boxed{\text{proj. var.}}$

upper-half plane

$\cdot i\infty$ ← near $i\infty$: $\Gamma\backslash D$ looks like





(lose cohomology classes if compactify into)

BS-cpt