Title: Gross-Zagier Formula: Why Is It Right?

Speaker: Shou-Wu Zhang

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What else should be right?

Algebraic cycles

$X/_{\mathbb{Q}}$ projective smooth, connected (not necessarily geometrically connected)

$n = \dim X$

$0 \le i \le n$

$$CH^i(X) = \frac{\overset{\sum n_i y_i}{\bigoplus_{Y \hookrightarrow X \text{ integ. subvar. of cod}=i} \mathbb{Z}}}{\left\langle \operatorname{div}_Z f \;\middle|\; \begin{array}{l} Z \hookrightarrow X \text{ integral subvar of cod}=i-1 \\ f \in \mathbb{Q}(Z)^* \end{array} \right\rangle}$$

<u>Conj</u> $CH^i(X)$ is finitely generated.

$i=1$ $\quad CH^1(X) = \mathrm{Pic}(X)(\mathbb{Q})$.

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Pic}^0(X)_{(\mathbb{Q})} & \to & \mathrm{Pic}(X)_{(\mathbb{Q})} & \to & NS(X) & \to & 0 \\ & & & & & \downarrow & & \\ & & & & & H^2_{\text{ét}}(X, \mathbb{Q}_\ell) & & \end{array}$$

<u>Conj is open</u> for $i > 1$.

Two step filtration

$$CH^i(X)_0 \hookrightarrow CH^i(X) \xrightarrow{cl} H^{2i}_{\acute{e}t}(\overline{X}, \mathbb{Q}_\ell(i))$$

cycles of homologically trivial $\qquad \overline{X} = X \otimes \overline{\mathbb{Q}}$

$C^i(X) = \mathrm{image}(cl)$

Conj. (Tate's conj.)

$$C^i(X) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} H^{2i}_{\acute{e}t}(\overline{X}, \mathbb{Q}_\ell(i))^{Gal(\overline{\mathbb{Q}}/\mathbb{Q})}$$

$$CH^i(X)_0 \overset{AJ}{\hookrightarrow} H^1_f(\mathbb{Q}, H^{2i-1}(\overline{X}, \mathbb{Q}_\ell(i)))$$

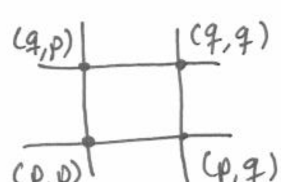
Conj. (Beilinson-Bloch)

$AJ \otimes \mathbb{Q}_\ell$ is an isomorphism.

Examples of cycles in $CH^i(X)_0$

① $C/\mathbb{Q}$ curve $\qquad p, q$ points in C

$C \times C$



$c_{p,q} := (p,p) + (q,q) - (p,q) - (q,p) \in CH^2(C \times C)_0$

$Aj(c_{p,q}) = 0 \underset{\text{conj.}}{\Longrightarrow} c_{p,q}$ torsion

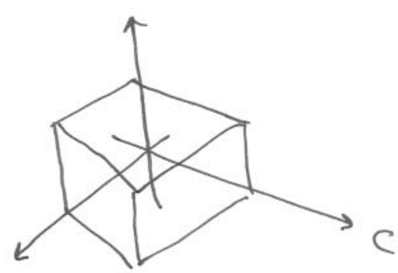
Ceresa cycle

$$C \hookrightarrow Jac(C)_g^{\circlearrowleft [-1]}$$

$\alpha \in H^{2g-2}(Jac)$ $\qquad$ $\underline{[-1]}c - c$ homologically trivial

$$\int_{[-1]c-c} \alpha = 0$$

$$\begin{array}{ccc} C & C & C \\ \Psi & \Psi & \Psi \\ p & p & p \end{array}$$



3 Gross-Schoen cycle

$$Z = \{(x,x,x) \mid x \in C\}$$
$$- \{(x,x,p)\} - \{(x,p,x)\} - \{(p,x,x)\}$$
$$+ \{(x,p,p)\} - \{(p,x,p)\} - \{(p,p,x)\}$$

$$\alpha \in H^2(C \otimes C, \mathbb{C})$$

$$\alpha = \bigoplus \alpha_{ij}$$

$$\int_Z \alpha = 0$$

Height pairing

$$i+j = n+1, \quad n = \dim X$$

$$\langle \ \ \rangle_{BB} : CH^i(X)_0 \otimes CH^j(X)_0 \longrightarrow \mathbb{R}$$

$i=j$ $\boxed{n = 2i-1}$

Beilinso-Bloch, non-degenerate

Rmk: By hard Lefschetz reduce to the case $(-1)^{i-1}\langle \ \ \rangle_{BB}$ positive definite

Beilinson-Bloch

(as refined by Bloch-Kato, Fontaine—Perrin-Riou)

generalization of BSD

$$\operatorname{rk} CH^i(X)_0 = \operatorname{ord}_{s=1} L(H^{2i-1}(\bar{X}, \mathbb{Q}_\ell(i)), s)$$

leading term identity

$i+j = n+1$

$H^{2i-1}(\bar{X}, \mathbb{Q}_\ell(i))$ is self-dual

$L(H^{2i-1}(\bar{X}, \mathbb{Q}_\ell(i)), s)$ self-dual

with a sign $\varepsilon = \pm 1$

$$\varepsilon = -1 \Rightarrow \mathrm{rk}\, CH^i(X)_0 \neq 0$$

$$\underset{n}{X} \underset{\mathrm{cod}=1}{\hookleftarrow} \underset{n-1}{Y}$$

$$\underset{n}{Y} \hookrightarrow \underbrace{X \times Y}_{2n-1}$$

Another triple product

morphism $/\mathbb{Q}$ $\quad Y \xrightarrow{\phi} \underset{\text{curve}}{\underset{\shortparallel}{C}}$ $\qquad Y \longrightarrow Y \times Y \times C$

$\qquad\qquad\qquad\qquad\qquad\qquad y \longmapsto (y, y, \phi(y))$

$$E = \begin{cases} \mathbb{Q} \\ \text{imaginary quadratic field} \end{cases}$$

$$\left(V_{/E}, (\ ,\ )\right) = \begin{matrix} \text{orthogonal of sig } (n,2) \\ \text{unitary of sig } (n,1) \end{matrix}$$

$G = U(V, (\ ,\ ))$ reductive gp $/\mathbb{Q}$

$\rightsquigarrow X_K$ = Shimura variety of dim $= n$

$K \hookrightarrow G(\hat{\mathbb{Q}})$
open compact
smooth compactification

$e \in V, \quad \langle e, e \rangle > 0$

$$W = (\mathbb{Q} e)^{\perp} = \begin{cases} (n-1, 2) \\ (n-1, 1) \end{cases}$$

Y_K = Shimura variety defined by $H = U((W, c))$

$$K_H = K \cap H(\hat{\mathbb{Q}})$$

$$Y_K = Y_K \times X_K = Z$$

<u>Example</u>: $n=1$ $\quad X_K$ = Shimura curve$/\mathbb{Q}$

$Y_K = CM\ pt = \mathrm{Spec}(L)$

$L/\mathbb{Q}$ number field

<u>$n=2$</u> <u>orthogonal case</u>

<u>Y_K</u> = <u>Shimura</u> curve $\quad X_K$ = Hilbert modular surface ($F/\mathbb{Q}$)

$= Y_K \times Y_K$

<u>$n=2$</u> <u>unitary case</u>

Y_K = Shimura curve $\quad X_K$ = Picard surface

$$Y_K \hookrightarrow Y_K \times Y_K$$

Gross-Kudla (Gross-Kudla-Schoen cycles)

In these cases Y_K do have a canonical decomposition into a sum

Y_K° and Eisenstein cycles

Homologically trivial cycle

<u>Arithmetic</u> Gan-Gross-Prasad

($E/$elliptic curve $\Pi = \mathrm{Hom}(X, E)$) $\mathbb{G} = G \times H$

Start with an irreducible rep of

$\mathbb{G}(\hat{\mathbb{Q}}) \times H(\hat{\mathbb{Q}})$

$$\Pi = \pi \times \sigma$$

$$Z_K = X_K \times Y_K \qquad Z = \varprojlim Z_K$$

$$CH^n(Z) \otimes \mathbb{Q} = \varinjlim_K CH^n(Z_K) \otimes \mathbb{Q}$$

$$MW(\Pi) = \mathrm{Hom}_{G(\hat{\mathbb{Q}})}(\Pi, CH^n(Z)_{\mathbb{Q}})$$

Conj: $\mathrm{rk}\ MW(\Pi) = \mathrm{ord}_{s=\frac{1}{2}} L(\Pi, s)$ – center

$$\{Y_K\}_K \in \varprojlim CH^n(Z_K)$$

Allow us to define a periodic integral

$$\begin{array}{ccl} \Pi & \longrightarrow & MW(\Pi) \\ \Psi & & \\ f & \longmapsto & P_H(f) := \int_Y f(y)\,dy \in MW(\Pi) \end{array}$$

Ingredients

① $P_H \in \mathrm{Hom}_{H(\hat{\mathbb{Q}})}(\Pi, \mathbb{C}) \otimes MW(\Pi)$

$$/\!/$$

$\prod_{p<\infty} \mathrm{Hom}_{H_p(\mathbb{Q}_p)}(\Pi_p, \mathbb{C})$ — $\dim \leq 1$

Aizenbud, Gourevitch

Rallis, Schiffmann

Sun, Zhu

② There is a distinguished generator α

$$\mathrm{Hom}_H(\Pi, \mathbb{C}) \otimes \mathrm{Hom}(\hat{\Pi}, \mathbb{C})$$

given by integration of matrix coefficients

Sakellaridis - Venkatesh

Conjecture (Thm?)

GGP $\quad$ W

If Arthur parameter ϕ of Π has trivial centralizer in $\hat{G}$

then there is a criterion of ε-factor for $\mathrm{Hom}_H(\Pi, \mathbb{C}) \neq 0$.

Combine all these

$\Rightarrow \exists\, c \in \mathbb{C}$ such that for any $f_1 \in \Pi$, $f_2 \in \tilde{\Pi}$

$$\langle P_H(f_1), P_H(f_2) \rangle_{BB} = c \cdot \alpha(f_1, f_2)$$

$$\langle\ \ \rangle_{BB} : MW(\Pi) \otimes MW(\tilde{\Pi}) \longrightarrow \mathbb{C}$$

Conjecture (GGP, refined by Wei Zhang)

$$P_H(f) \not\equiv 0 \text{ iff } L'(\Pi, \tfrac{1}{2}) \neq 0$$

and $c = L'(\Pi, \tfrac{1}{2})$

$n=1$ GZ YZZ

$n=2$ orthogonal case GK YZZ $\underbrace{\text{(special cases)}}_{\text{Weil rep and Kudla generating function}}$

$n>1$ unitary case

Wei Zhang proposed a relative trace formula approach (AFL, ST, ...)

AFL $n=1$
$n=2$