Title: An Example-Based Introduction to Shimura Varieties and Their Compactifications

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$$\Gamma \backslash D \hookrightarrow \underset{\text{proj. var. normal}}{(\Gamma \backslash D)^{\min}}$$ Satake Baily-Borel

idea: just like in the modular curve case.

$$\Gamma \backslash \mathfrak{h} \hookrightarrow \Gamma \backslash (\mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q}))$$
(with suitable topology)

$$\mathbb{P}^1(\mathbb{Q}) = \underbrace{SL_2(\mathbb{Q})}_{}\cdot \infty$$

in proj coord. $\infty = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

elements of $SL_2(\mathbb{Q})$ preserving $\begin{pmatrix} * \\ 0 \end{pmatrix}$ are of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

What we need in general are the <u>max.</u> rational parabolic subgroups of the group.

(parabolic P means G/P proj. var.)

"reduction theory"
($\Rightarrow$ only "rat'l boundary components" matter)

<u>Ex 1</u>: $\underbrace{Sp_4(\mathbb{Q})}_{\text{standard parabolics}} \curvearrowright \mathfrak{h}_2$

Sp_{2n} respects the pairing def'd by $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

<u>Borel</u>: $\left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & \\ & & * & * \end{pmatrix} \right\} \longrightarrow \begin{pmatrix} * \\ 0 \\ 0 \\ 0 \end{pmatrix} \subset \begin{pmatrix} * \\ * \\ 0 \\ 0 \end{pmatrix}$

<u>max ones</u> are $Sp_4(\mathbb{Q})$ conj.s of the following

<u>Siegel</u>: $\left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix} \right\}$

$\curvearrowright \begin{pmatrix} * \\ * \\ 0 \\ 0 \end{pmatrix}$

<u>Klingen</u>: $\left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & \\ & * & * & * \end{pmatrix} \right\}$

$\curvearrowright \begin{pmatrix} * \\ 0 \\ 0 \\ 0 \end{pmatrix}$

General principle
(for "classical groups" def'd by pairings)
each rat'l parabolic preserves some flag of "isotropic" (rat'l) subspaces.

$Sp_{2n}(\mathbb{Q})$: have max rat'l parabolics

Conj. to
$$\begin{pmatrix} \boxed{*} & * & * & * \\ & * & * & * \\ & & \boxed{*} & \\ & * & * & * \end{pmatrix}$$
unipotent radical

size $r \times r$ GL_r

each size $n-r$

$\rightsquigarrow Sp_{2(n-r)} \curvearrowright \mathfrak{h}_{n-r}$

Levi: $\underbrace{GL_r}_{\tilde{G}_\ell} \times \underbrace{Sp_{2(n-r)}}_{G_h}$

Herm. part

the "boundary component" of coord
$$\begin{pmatrix} \begin{array}{c} r \end{array}\boxed{\infty_r} & \\ & \begin{array}{c} n-r \end{array}\boxed{Z_r} \end{pmatrix} \sim \begin{pmatrix} 1 & Z_r \\ 0 & 1 \end{pmatrix}$$

$\cong \mathfrak{h}_{n-r}$ $\qquad Z_r \in \mathfrak{h}_{n-r}$ $\qquad$ in proj. coord.

$$\mathfrak{h}_n^* = \mathfrak{h} \cup Sp_{2n}(\mathbb{Q})\mathfrak{h}_{n-1} \leftarrow \text{dim } \tfrac{1}{2}n(n-1)$$
$$\cup\, Sp_{2n}(\mathbb{Q})\mathfrak{h}_{n-2}$$
$$\vdots$$
$$\cup\, Sp_{2n}(\mathbb{Q})\mathfrak{h}_0 \leftarrow \text{dim } 0$$

dim $\frac{1}{2}n(n+1)$ w/ Satake top.

$$(\Gamma \backslash \mathfrak{h}_n)^{min} = \Gamma \backslash \mathfrak{h}_n^*$$
$$\cup$$
$$\Gamma \backslash \mathfrak{h}$$

Stratified by images of "smaller $\Gamma \backslash \mathfrak{h}_n$"

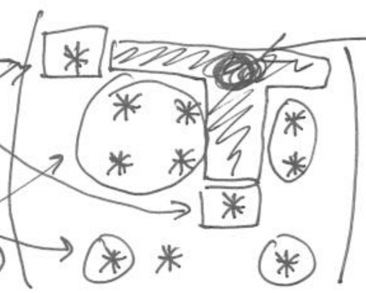
<u>EX 2:</u> $\quad U_{p,q}(\mathbb{Q}) \curvearrowright \mathfrak{h}_{p,q}$

(def'd by some Herm. pairing over imag. quad. field K with signature (p,q) at ∞)

$$\begin{pmatrix} & & I_q \\ & S & \\ -I_q & & \end{pmatrix} \qquad \left\{ \begin{array}{r} r\,[\\ q-r\,[\\ p-q\,[\\ r\,[\\ q-r\,[\end{array} \begin{pmatrix} * \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \leftarrow \text{isotropic rat'l}$$

max parabolic are conj. to

$GL_r(K)$



two step vector group (unipotent radical)

$U_{p-r,q-r}(\mathbb{Q})$ def'd by $\begin{pmatrix} & & I_{q-r} \\ & S & \\ -I_{q-r} & & \end{pmatrix}$

$$\mathfrak{h}^*_{p,q} = \mathfrak{h}_{p,q} \bigcup_{0 \le r \le q} \left(U_{p,q}(\mathbb{Q})\, \mathfrak{h}_{p-r,q-r} \right)$$

$$(\Gamma \backslash \mathfrak{h}_{p,q})^* = \Gamma \backslash \mathfrak{h}^*_{p,q}$$

<u>Ex 0'</u> (revisited)

For $G(\mathbb{R}) = SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$

if $G(\mathbb{Q}) = SL_2(\mathbb{Q}) \times SL_2(\mathbb{Q})$

then $G(\overline{\mathbb{Q}}) \curvearrowright \mathfrak{h} \times \mathfrak{h}$

$$(\mathfrak{h} \times \mathfrak{h})^* = \mathfrak{h}^* \times \mathfrak{h}^*$$

$$= \mathfrak{h} \times \mathfrak{h} \cup G(\mathbb{Q})(\infty \times \mathfrak{h})$$

$$\cup\ G(\mathbb{Q})(\mathfrak{h} \times \infty)$$

$$\cup\ G(\mathbb{Q})(\infty \times \infty)$$

If $F =$ tot. real quad $/\mathbb{Q}$

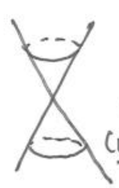
$$G(\mathbb{Q}) = SL_2(F) \curvearrowright \mathfrak{h} \times \mathfrak{h}$$

$$\underset{\text{dim } 2}{(\mathfrak{h} \times \mathfrak{h})^*} = \mathfrak{h} \times \mathfrak{h} \cup \underbrace{G(\mathbb{Q})\text{“}\infty\text{”}}_{\text{only one orbit}}$$

$\boxed{\text{dim } 0}$

$$(\Gamma \backslash (\mathfrak{h} \times \mathfrak{h}))^{mn} = \text{Hilbert modular surface}$$

“cusps” = 0-dim



normal (not regular)

smooth

Ex 3:

$$\mathfrak{h}^*_{SO^*_{2n}} = \mathfrak{h}_{SO^*_{2n}} \cup \bigcup_{0 < r \le [\frac{n}{2}]} G(\mathbb{Q}) \cdot \mathfrak{h}_{SO^*_{2n-4r}}$$

dim $= \frac{1}{2} n(n-1)$

Ex 4: $\mathfrak{h}^*_{SO_{p,2}} = \mathfrak{h}_{SO_{p,2}}$

$UG(\mathbb{Q}) \cdot \mathfrak{h}_1$

$UG(\mathbb{Q}) \cdot \mathfrak{h}_0$

Ex 5: $\mathfrak{h}^*_{E_7} = \mathfrak{h}_{E_7}$

dim 27

$UG(\mathbb{Q}) \cdot \mathfrak{h}^{(+)}_{SO_{10,2}}$ ← dim 10

$UG(\mathbb{Q}) \cdot \mathfrak{h}_1$ ← dim 1

$UG(\mathbb{Q}) \cdot \mathfrak{h}_0$ ← dim 0

Ex 5': $\mathfrak{h}^*_{E_6} = \mathfrak{h}_{E_6}$

dim 16

$UG(\mathbb{Q}) \cdot \mathfrak{h}_{5,1}$ ← dim 5

$UG(\mathbb{Q}) \cdot \mathfrak{h}_0$