Title: Cohomology of Arithmetic Groups

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Towers

$$T = \{ M \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots$$

also $\{ \Gamma \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots$

Def (completed (co)homology)

$$\tilde{H}_d(T, \mathbb{F}_p) = \varprojlim H_d(M_i, \mathbb{F}_p)$$

$$\text{or } \varprojlim H_d(\Gamma_i, \mathbb{F}_p)$$

$$\tilde{H}_d(T, \mathbb{Z}_p) = \varprojlim H_d(M_i, \mathbb{Z}_p)$$

$$\tilde{H}^d(T, \mathbb{F}_p) = \varinjlim H^d(M_i, \mathbb{F}_p)$$

$$\tilde{H}^d(T, \mathbb{Z}/p^r\mathbb{Z}) = \varinjlim H^d(M_i, \mathbb{Z}/p^r\mathbb{Z})$$

$$\tilde{H}^d(T, \mathbb{Z}_p) = \varprojlim \tilde{H}^d(T, \mathbb{Z}/p^r\mathbb{Z}$$

$X = \coprod \Gamma \backslash \mathcal{H}$ contractible

$\uparrow G(\mathbb{R})/K(\mathbb{R})$

$H^*(\Gamma\backslash\mathcal{H}) = H^*(\Gamma)$

* $S^1 = M$ covers given multiplication by p^n

* $X \leftarrow X(p) \leftarrow X(p^2) \leftarrow X(p^3) \leftarrow \cdots$

$X =$ modular curve, Shimura curve, Shimura variety

Shimura manifold, ...

* $\Gamma =$ arith group. $\Gamma \supseteq \Gamma(p) \supseteq \Gamma(p^2) \supseteq \cdots$

* Γ lattice in a parabolic subgroup

$H_0(S^1, \mathbb{Z}_p) = \mathbb{Z}_p$ $\quad H_1(S^1; \mathbb{Z}_p) = \mathbb{Z}_p$

$$\boxed{\begin{aligned}\tilde{H}_0 &= \mathbb{Z}_p \\ \tilde{H}_1 &= 0\end{aligned}}$$

$0 \leftarrow \bigcirc \leftarrow \bigcirc \leftarrow \cdots$

maps on $H_0 \cong$

$\tilde{H}_0(\mathbb{Z}_p) = \mathbb{Z}_p$ $\quad \tilde{H}_1(\mathbb{Z}_p) = (\mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \cdots)$

Modular curve

Connected: $\tilde{H}_0(\mathbb{Z}_p) = \mathbb{Z}_p$ $\quad$ module for $\mathbb{Z}_p[[GL_2\mathbb{Z}_p]]$

Non-connected: $\tilde{H}_0(\mathbb{Z}_p) = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$

$\tilde{H}_2 = 0$; $\tilde{H}_1$ positive rank over $\mathbb{Z}_p[\![GL_2\mathbb{Z}_p]\!]$

$X =$ Shimura curve $(= \Gamma \backslash \mathbb{H})$

$H_2(X, \mathbb{Z}_p) = \mathbb{Z}_p$

$\tilde{H}_2(\mathbb{Z}_p) = 0$

$\Gamma_N = SL_N \mathbb{Z}$ $\qquad N \geq 3$

$\Gamma_N \supseteq \Gamma_N(p) \supseteq \dots \supseteq \Gamma_N(p^k) \supseteq \dots$

$H_1(\Gamma, \mathbb{Z}) = \Gamma^{ab}$

$H_1(\Gamma, \mathbb{Z}_p) = \Gamma^{ab} \otimes \mathbb{Z}_p$

Analysis: $H_1(\Gamma(p^m), \mathbb{R}) = 0$

Say nothing about $H_1(\Gamma(p^m), \mathbb{F}_p)$

are there some obvious abelian quotients?

$$\Gamma_N(p^m) \longrightarrow \Gamma_N(p^m) / \Gamma_N(p^{2m})$$

<u>Thm</u> (Mennicke, Bass-Milnor-Serre)
<u>Any</u> finite index sub of Γ_N is congruence.
$\Rightarrow H_1(\Gamma_N(p^m), \mathbb{Z}_p) \cong \Gamma_N(p^m) / \Gamma(p^{2m})$

$$\uparrow$$

$$H_1(\Gamma_N(p^{2m}), \mathbb{Z}_p) \qquad \therefore\ \tilde{H}_1 = 0$$

<u>$N \geq 3$</u> $D/_{\mathbb{Q}}$ div. alg. split at ∞.

$$\text{inv} = \frac{1}{N}, -\frac{1}{N}, l_1, l_2 \neq p$$

$\Gamma :=$ $B' \subseteq D$ int. norm $= 1$

$$B' \longrightarrow SL_N \mathbb{Z}_p$$

$$\Gamma \supseteq \Gamma(p) \supseteq \Gamma(p^2) \supseteq \cdots$$

<u>Remark</u>: groups of rank $1\frac{1}{2}$

$$Sp(n,1),\ U(m,1),\ F_{4(-20)} \qquad n \geq 2$$

<u>Conj</u>: in these cases, $\tilde{H}_1$ vanishes/is finite.

$\tilde{H}_1$ of $\Gamma = SL_N \mathcal{O}_F$ $(N \geq 3)$

is <u>finite</u>.

$$\Gamma = GL_N \mathcal{O}_F \text{ "="} SL_N \mathcal{O}_F \times \mathcal{O}_F^\times$$

Compute $\tilde{H}_1$ $\quad \Gamma = \mathcal{O}_F^\times = \mathbb{Z}^{r_1+r_2-1} \oplus H_F$

$$0 \longrightarrow \underset{}{\overset{\Gamma(p^m)}{\overset{\shortparallel}{K}}} \longrightarrow \mathcal{O}_F^\times \longrightarrow (\mathcal{O}_F/_{p^m})^\times$$

$H^1(\Gamma(p^m), \mathbb{Z}_p)$

$$\mathcal{O}_F^\times \otimes \mathbb{Z}_p \longrightarrow \prod_{\mathfrak{p} | p} \mathcal{O}_{F,\mathfrak{p}}^\times \otimes \mathbb{Z}_p$$

$\tilde{H}_1$ = kernel = finite

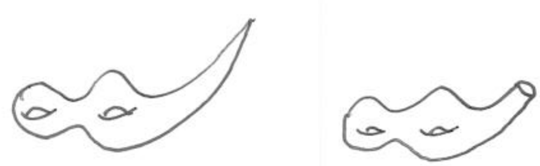
$\Longleftrightarrow$ Leopoldt's conjecture is true.

$\tilde{H}_1$ of $GL_N \mathcal{O}_F$ is f.g. $\mathbb{Z}_p$-mod

$\tilde{H}_d$ of $GL_N \mathcal{O}_F$ is f.g. $\mathbb{Z}_p$-mod

independent of N $(N \gg d)$

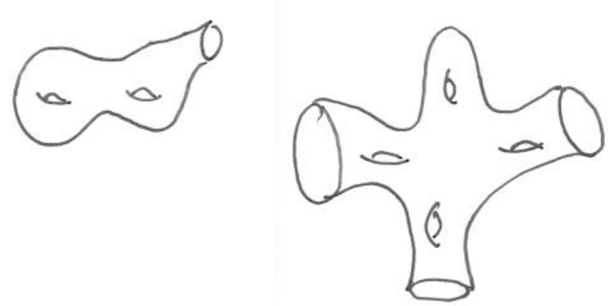
Boundaries



$X \hookrightarrow X^{BS}$

$$H^{n-1}(\partial X^{BS}) \longrightarrow H^n_c(X) \longrightarrow H^n(X^{BS})$$

$P'(\mathbb{Q}_p)$



$\mathrm{Ind}_{P(\mathbb{Z}_p)}^{GL_2 \mathbb{Z}_p} \mathbb{Z}_p$

$SL_2\mathbb{Z}[p] \longrightarrow SL_2\mathbb{C}/SU_2\mathbb{C}$

hyperbolic 3-space

$\tilde{H}_0 = \mathbb{Z}_p$

$\tilde{H}_1$ = complicated

$\tilde{H}_2$ = ?? 0

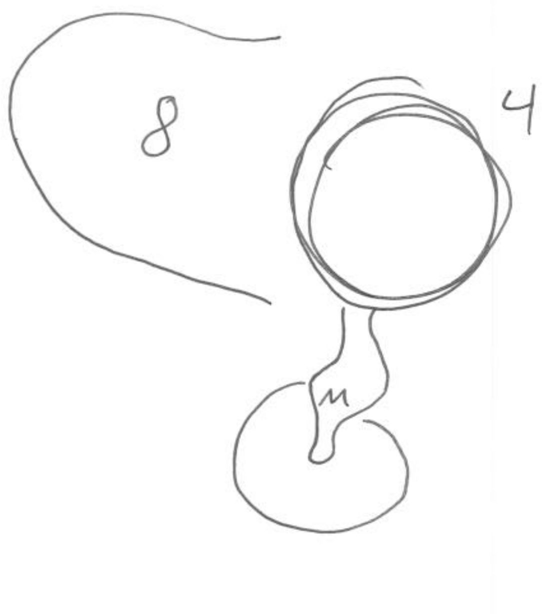
$U(2,2)$ $\quad$ $\mathbb{Q}(i)$

dim = 4 = 8 real dimensions

$P = MN$

M $\quad$ $SL_2(\mathbb{Z}[i])$ $\quad$ dim = 3

dim $N = 4$



$\mathrm{Ind}_P^G \tilde{H}(SL_2\mathbb{Z}[i])$

$$\tilde{H}^i(X, \mathbb{F}_p) \otimes \mathcal{O}_{\mathbb{C}_p}/p$$

$$= H^i(\mathcal{X}, \mathcal{O}^+/p) = 0 \quad \text{if } i > n$$

$\dim \mathcal{X} = n$ (complex)

$2n$ real dim