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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Sean Howe **Email/Phone:** seanpkh@gmail.com

Speaker's Name: Marie-France Vigneras

Talk Title: Parabolically induced mod p representations of reductive p-adic groups

Date: 12 / 1 / 2014 **Time:** 9 : 30 am / pm (circle one)

List 6-12 key words for the talk: Modular representations, supersingular and supercuspidal, parabolic induction

Please summarize the lecture in 5 or fewer sentences: Explains recent work on non-supercuspidal representations of p-adic reductive groups; gives a classification of these using supercuspidal admissible triples, and explains the role of supersingular representations in the proof (a key step is supersingular = supercuspidal).

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - <u>Computer Presentations</u>: Obtain a copy of their presentation
  - <u>Overhead</u>: Obtain a copy or use the originals and scan them
  - <u>Blackboard</u>: Take blackboard notes in black or blue **PEN.** We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - <u>Handouts</u>: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack <u>with this completed sheet on top</u> and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
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Abe-Henniart-Herzig-V

Available on home page of one of them, soon on arXiv.

$F/\mathbb{Q}_p$ a Fin. ext.
or $F = \mathbb{F}_p((T))$

$\underline{G}$ reductive connected group $/F$

$G = \underline{G}(F)$, $T \subset B = ZU$

$\uparrow$ minimal parabolic

C an alg. closed field of char. p.

Want: Classify irred. representations of G over C
(continuous)

Representation π of G admissible if
$H \subseteq G$ open compact $\Rightarrow \dim \pi^H < \infty$

Irr_C^{adm} — supercuspidal $\leftarrow$ "Blackhole w/strong power of attraction"
— non-supercuspidal

Classification of the non-supercuspidal irr. adm. rep's:

Barthel-Livné $GL_2(F)$ started for all F

$\downarrow$

Herzig jumped to $GL_n(F)$ for all n, $F \supset \mathbb{Q}_p$

$\downarrow$

Abe split $F \supset \mathbb{Q}_p$

Abe-Henniart-Herzig-V general G, F

$$P \supset B \qquad \operatorname{Ind}_P^G : \underbrace{\mathrm{Mod}_C M}_{\text{Rep.'s } (\tau, W)} \longrightarrow \mathrm{Mod}_C(G)$$

$P = MN$, $N \supseteq Z$

$\operatorname{Ind}_P^G W =$
$\{F : G \to W \mid F(mngk) = \tau(m)F(g)$ for $K \in$ open compact s.g. $\}$

$G \circlearrowleft$ by right translation.

<u>Prop</u> Parabolic induction is faithful w/a left, right adjoint and respects admissibility.

<u>Def</u> $\pi \in \mathrm{Irr}_C^{adm}(G)$ is supercuspidal if it is not a subquotient of $\operatorname{Ind}_P^G \tau$ for $P \neq G$, $\tau \in \mathrm{Irr}_C^{adm}(M)$

(Note: it is important that τ here is irreducible, not true otherwise)

<u>Supercuspidal admissible triples:</u> (P, σ, Q), $B \subseteq P = MN$

$\sigma \in \mathrm{Irr}_C^{adm}(M)$ supercuspidal

$$P \subset Q \subset \underbrace{P(\sigma)}_{\substack{\text{Biggest subgroup}\\ \text{where } \sigma \text{ can be}\\ \text{extended trivially on } N}}$$

<u>Generalized Steinberg rep.:</u> $St_Q^{P(\sigma)}(\sigma) := \dfrac{\operatorname{Ind}_Q^{P(\sigma)} e(\sigma)}{\sum_{Q \subsetneq Q' \subseteq P(\sigma)} \operatorname{Ind}_{Q'}^{P(\sigma)} e(\sigma)}$ ← e = extension

$$I(P,\sigma,Q) = \mathrm{Ind}_{P(\sigma)}^{G} \, St_Q^{P(\sigma)}(\sigma)$$

Thm Bijection $(P,\sigma,Q) \leftrightarrow I(P,\sigma,Q)$

$$\left\{ \begin{array}{c} \text{supercuspidal} \\ \text{admissible triples} \end{array} \right\} \xrightarrow{\sim} \mathrm{Irr}_C^{adm}(G)$$

First application ① $\mathrm{Ind}_P^G \tau$, $\tau \in \mathrm{Irr}_C^{adm}(M)$
is finite length, with irreducible subquotients admissible and occurring w/multiplicity one.
(Remark: no intertwiners, different from complex case)

② Supercuspidal support — deduced from classification.

Second application Thm Supercuspidal = supersingular

Satake homomorphism: (heart of theorem)
K open compact subgroup "special parahoric"

$$G = BK$$

$$B \cap K = Z_0 U_0 \qquad P \cap K = M_0 N_0$$

$$C[K \backslash G / K], * \hookrightarrow C[Z/Z_0], *$$

Spherical Hecke algebra

$$\xrightarrow{\sim} C[Z^+/Z_0], *$$

$V \in \mathrm{Irr}_C(K)$ compact induction $\mathrm{ind}_K^G V = \left\{ f: G \to V \text{ w/ compact support, } f(kg) = k f(g) \right\}$

Spherical Hecke algebra (Generalization)

$$= \mathrm{End}_{CG}(\mathrm{Ind}_K^G V) = \mathcal{H}(G, K, V)$$

$$\{F : G \to \mathrm{End}_C V,\ F(k_1 g k_2) = k_1 F(g) k_2 \}$$
with compact support

$$\mathcal{H}(G,K,V) \longleftrightarrow \mathcal{H}(Z, Z_0, \underbrace{V_{U_0}}_{\psi_V})$$

V_{U_0} 1-dim'l

$Z_0 \circlearrowright$ via ψ_V.

$$\mathcal{H}(G,K,V) \xrightarrow{\sim} \mathcal{H}^+(Z^+, Z_0, \psi_V)$$

$$\text{Center} = Z(Z^+, Z_0, \psi_V)$$

Z/Z_0 commutative (Haines)

Z/Z_U not commutative in general.

pro-p sylow, so this is why we move to center

$$\chi \otimes_{Z(Z^+, Z_0, \psi_V)} \mathrm{ind}_K^G V \qquad \chi : Z(Z^+, Z_0, \psi_V) \to C$$

character

(Cut $\mathrm{ind}_K^G V$ into small pieces to study)

$\pi \in \mathrm{Irr}_C^{adm}(G)$

<u>Def</u> V is a weight of π and χ is an eigenvalue of $Z(G,K,V)$ if π is a quotient of $\chi \otimes \mathrm{ind}_K^G V$

(sometimes called Serre weight)

$$\begin{tikzcd} Z(G,K,V) \arrow[rr, hook] \arrow[dr, hook] & & Z(Z, Z_0, V_{U_0}) \\ & Z(M, M_0, V_{N_0}) \arrow[ur] & \end{tikzcd}$$

χ is called supersingular if it does not extend to

$Z(M, M_0, V_{N_0})$ for all $P = MN \neq G$

π is supersingular if for all weights V in π, eigenvalues χ in π, χ is supersingular.

(Remark: A priori depends on choice of K, but after proving supersingular = supercuspidal)

Going from compact induction to parabolic induction

<u>Ex</u> $C[K \backslash G] \hookrightarrow \mathrm{Ind}_B^G \, C[Z_0 \backslash Z]$ Equivariant for $\mathcal{H}(G, K, \psi_V)$

($C[K \backslash G]$: compact ind. of trivial)

$$\mathrm{ind}_K^G V \hookrightarrow \mathrm{Ind}_B^G \, \mathrm{ind}_{Z_0}^Z \underbrace{V_{U_0}}_{= \psi_V}$$

V "regular" (will not define) $\mathrm{ind}_K^G V \hookrightarrow \mathrm{Ind}_P^G \, \mathrm{ind}_{M_0}^M V_{N_0}$

$\chi \in \hat{Z}(M, M_0, V_{N_0})$ restriction of a char. of $Z(G,K,V)$

$$\chi \otimes \mathrm{ind}_K^G \simeq \chi \otimes \mathrm{Ind}_P^G$$

Change of weight

Some conditions so that $\longrightarrow$ $\chi \otimes_{Z(G,K,V)} \mathrm{ind}_K^G V \simeq \chi \otimes_{Z(G,K,V')} \mathrm{ind}_K^G V'$

$$Z(G,K,V') \cong Z(Z^+, Z_0, \Psi_V)$$

(assume $\Psi_V = \Psi_{V'}$ $\leftarrow$ Hecke depends only on this)

proved by using action of the pro-p Iwahori in a representation containing both, $\mathrm{Ind}_B^G \mathrm{ind}_{Z_0}^Z V_{U_0}$

$\mathcal{H}_C(G, I_U)$ $\quad I_U$ = pro-p radical of the Iwahori I

$$I_U \cap B_0 = Z_U U_0$$

$$\mathrm{Mod}_C(G) \xrightarrow{I_U\text{-inv.}} \mathrm{Mod}_{\mathcal{H}_C}(G, I_U)$$

left adjoint $\quad - \otimes C[I_U \backslash G]$

$V \in \mathrm{Mod}_C G$, $\neq 0$ $\Rightarrow$ $V^{I_U} \neq 0$

Classification of $\mathrm{Irr}\,\mathcal{H}_C(G, I_U)$

- supersingular
- non-supersingular

Difference between $GL_2(\mathbb{Q}_p)$ and $GL_2(F)$ -
in general Iwahori invariants of an irred. are not irred.

$$
\begin{tikzcd}
\mathrm{Mod}_c(G) \arrow[r, shift left, "I_U\text{-inv.}"] & \mathrm{Mod}\, \mathcal{H}_c(G, I_U) \arrow[l, shift left] \\
\mathrm{Mod}_c(M) \arrow[u, "\mathrm{Ind}_P^G"] \arrow[r, shift left, "I_{U,M}\text{-inv.}"] & \mathrm{Mod}\, \mathcal{H}_c(M, I_{U,M}) \arrow[l, shift left] \arrow[u, "\mathrm{Ind}_P^G"']
\end{tikzcd}
$$

commutes

Thm 1) Parabolic induction commutes w/ pro-p Iwahori functions and its left-adjoint.

2) (*) $\pi \in \mathrm{Irr}_c^{adm}(G)$ supersingular $\iff$ π^{I_U} has all its irred. subquotients supersingular.