MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Sean Howe **Email/Phone:** seanpkh@gmail.com

Speaker's Name: Matt Emerton

Talk Title: Moduli Stacks of Local Galois Representations

Date: 12 / 1 / 2014 **Time:** 11 : 00 am / pm (circle one)

List 6-12 key words for the talk: Crystalline representations, mod p representations, moduli of representations, algebraic stacks, reduction maps

Please summarize the lecture in 5 or fewer sentences: Motivates the construction of moduli stacks of crystalline and mod p representations and reduction maps using an example w/ 2-dimensional representations in low weight. Discusses general construction by using map from integral p-adic Hodge Theory data to Galois data (étale-φ modules) and taking scheme theoretic image. Concludes with theorem showing scheme theoretic image is a stack under verifiable hypotheses.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Joint w/ Toby Gee

Idea: To make moduli spaces of p-adic and mod p reps of G_K where $K/\mathbb{Q}_p$ is finite.

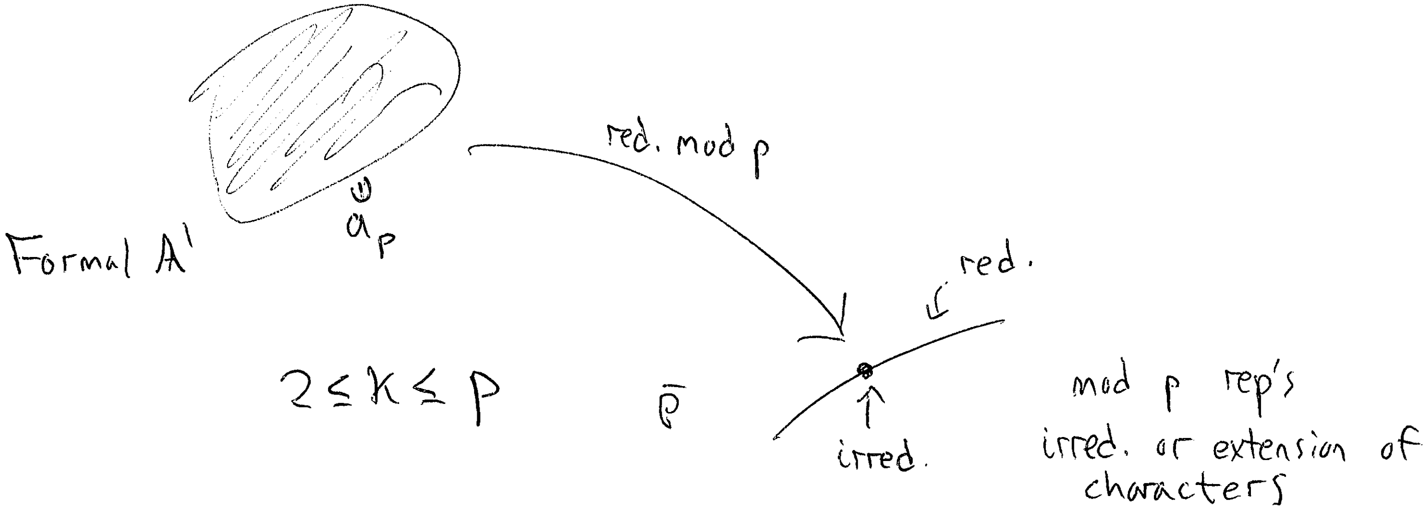
<u>Mazur, et al.</u>: Formal moduli of such rep.'s
We want algebraic moduli
(so ρ really varies in families)

<u>One motivation</u>: Reduction of crystalline Galois rep.'s

If $K=\mathbb{Q}_p$, $\rho: G_{\mathbb{Q}_p} \to GL_2(\overline{\mathbb{Q}_p})$ continuous, crystalline w/ Hodge-Tate weights $0, k-1$

$\det = \mathrm{cyclo}^{1-k}$

ρ is essentially classified by a_p = trace of crystalline Frobenius
(because if reducible then classifies semi-simplification)

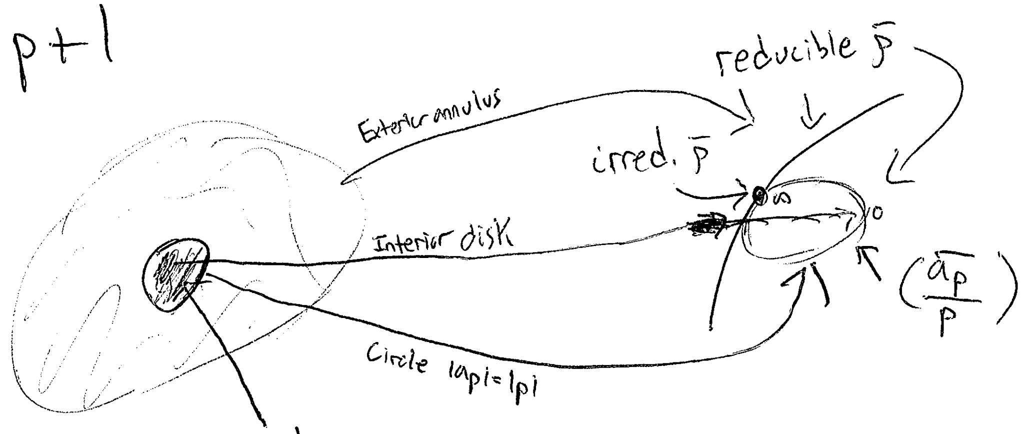


$2 \leq k \leq p$

$a_p \longmapsto \overline{a_p}$ is the reduction

a_p a unit $\Rightarrow \bar{\rho}$ reducible

$K = p+1$



Formal $\mathbb{A}^1/\mathbb{Z}_p$ blown up at a point

a_p $\quad$ $p | a_p$

Reduction map is a specialization on formal schemes

Goal: For any $K/\mathbb{Q}_p$ finite, type τ, integer h, to construct a formal stack $/\mathbb{Z}_p$ (top. of finite type), $\mathcal{X}^{\tau,h}$ with

$$\overline{\mathbb{Z}}_p\text{-points} \longleftrightarrow \rho : G_K \to GL_2(\overline{\mathbb{Z}}_p)$$

s.t. $\rho \otimes \mathbb{Q}_p$ is potentially semistable with weights in $[0,h]$ and type τ.

And also construct $\overline{\mathcal{X}}$, a f.t. alg. stack $/\mathbb{F}_p$ s.t. $\overline{\mathbb{F}}_p$-points of $\overline{\mathcal{X}} \longleftrightarrow \bar{\rho} : G_K \to GL_n(\overline{\mathbb{F}}_p)$

And give an embedding

$$(\overline{\mathcal{X}^{\tau,h}})_{red} \hookrightarrow \overline{\mathcal{X}}$$

with image a union of components s.t. the specialization map gives

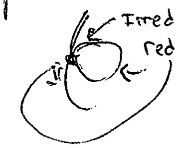
$$\mathcal{X}^{\tau,h}(\overline{\mathbb{Z}}_p) \xrightarrow{sp} (\overline{\mathcal{X}}^{\tau,h})_{red}(\overline{\mathbb{F}}_p) \to \overline{\mathcal{X}}(\overline{\mathbb{F}}_p)$$

$$\rho \longmapsto \bar{\rho}$$

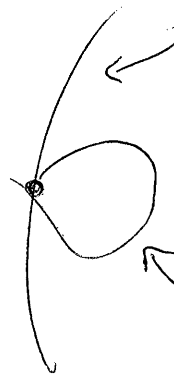
In $K = p+1$

Two components correspond to two weights
Meet at points w/companion form

Morally looking for something like a representation variety,
but e.g. in a rep. variety ~~ir~~reducibles specialize to reducibles,
whereas here we see that the irreducible point
is a specialization of reducibles



$\begin{pmatrix} ur & * \\ 0 & ur \cdot cyclo^{1-k} \end{pmatrix} \longleftrightarrow Sym^{k-2}\overline{\mathbb{F}}_p^{\,2}$
(weight)



$\overline{cyclo}^{-1} \otimes \begin{pmatrix} ur & * \\ 0 & ur \end{pmatrix} \leftrightarrow Sym^{k-2}\overline{\mathbb{F}}_p^{\,2} \otimes \det^{-1}$
(weight)

IF h is small, one can make components of $\mathfrak{X}$ as moduli spaces of Fontaine-LaFaille modules

└ defined by lin. alg. data, so can easily construct moduli (Daniel Le's thesis discusses these)

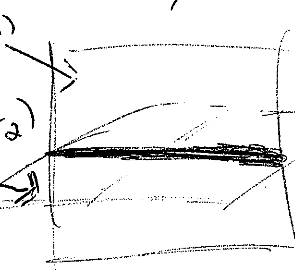
IF h is not small or $K/\mathbb{Q}_p$ is too ramified, this doesn't work.

Example: $K = \mathbb{Q}_p$, GL_2, $h = p-1$ det = trivial, then reducible $\bar{\rho}$ are in a family

$\begin{pmatrix} ur_\alpha & * \\ 0 & ur_{\alpha^{-1}} \end{pmatrix}$

$Ext(ur_\alpha, ur_{\alpha^{-1}})$

$Ext(ur_{\alpha^{-1}}, ur_\alpha)$

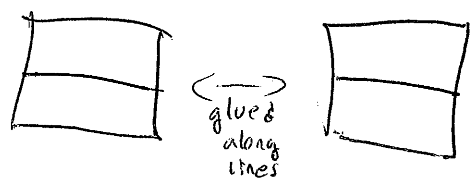


α-line, split $\bar{\rho} = ur_\alpha \oplus ur_{\alpha^{-1}}$
Two branches intersect along split $\bar{\rho}$

Companion form corresponds to this local crossing at this double line

Can compute Ext^1 using FL, gives two spaces

one for each Ext. But FL Functor is not faithful at this boundary weight, and thus joins these two sheets



In general, integral p-adic Hodge theory will produce spaces like this, so problem of constructing $\bar{\mathfrak{X}}$ is a problem of contracting these types of spaces.

Good progress towards construction of $\mathfrak{X}$, $\bar{\mathfrak{X}}$, etc.
Details worked out after replacing K with $K_\infty = \bigcup_{n \geq 1} K(\pi^{1/p^n})$
This is enough to handle the case $h \leq 1$ (Barsotti-Tate $\mathfrak{p}$, Finite Flat $\mathfrak{F}$)

To get general case must add $\hat{G}$-structure following Tong Liu.

Over K_∞, Fontaine-Breuil-Kisin give a nice integral p-adic HT, which Pappas-Rapoport put in moduli.

a fixed, d fixed

$$\mathcal{C}^{\leq h} \to \mathcal{R}$$

$$M \mapsto M[\tfrac{1}{u}]$$

moduli-stack of Breuil-Kisin modules of height $\leq h$

moduli stack of étale φ-modules

(M - $A((u))$-module locally free of rank d, $\varphi^*M \to M$)

over some $\mathbb{Z}/p^a$ algebra A

A-points are $M/_{A[[u]]\otimes W(k)}$

$\varphi^* M \to M$

coker killed by $E(u)^h$

The RHS corresponds by étale-φ theory to Galois rep.'s, so this generalizes FL theory

Believe that all $\bar{\rho}$ should appear as reductions of crystalline rep.'s for sufficiently large h.

So, want to build $\overline{\mathcal{X}}$ as the union of images of this map. In fact, should be image just for h sufficiently large (e.g. p^2 for GL_2)

Need to construct these images as objects in algebraic geometry

What we have:

$$\mathcal{C}^{\leq h} \longrightarrow R \text{ is proper}$$

alg. stack of fin type $/\mathbb{Z}/p^a$

huge
(ind. alg. stack, but too big to be a stack)

$\Delta : R \to R \times R$
is representable, finite type

Relativization of statement that aut. of a Galois rep is a subgroup of GL_d

R admits versal rings at each finite type point.

Setup: $\mathcal{X} \xrightarrow{\xi} \mathcal{F}$

alg. stack of f.t. $/S$

$\Delta : \mathcal{F} \to \mathcal{F} \times \mathcal{F}$
is representable by algebraic spaces locally of finite ~~type~~ presentation,
$\mathcal{F}$ has versal rings at f.t. points

Define "Scheme-theoretic image" of ξ, $\mathcal{Z} \hookrightarrow \mathcal{F}$

If A is locally Artinian, f.t. $/S$, $\mathcal{Z}(A) \subseteq \mathcal{F}(A)$ are the maps $\operatorname{Spec} A \to \mathcal{F}$ which factor
$\operatorname{Spec} A \to \operatorname{Spec} B \to \mathcal{F}$
where
- B is complete local Noetherian $/S$
- $\mathcal{X}_B \to \operatorname{Spec} B$ is sch. theoretically dominant

If T/S is finite type, $Z(T) \hookrightarrow F(T)$

is maps s.t. $\mathrm{Spec}\, \mathcal{O}_{T,t}/\mathfrak{m}_t^n \to F$

lies in Z $\forall n \geq 1$, all t a f.t. pt in T.

If T/S is affine, write $T = \varprojlim T_i$ where T_i f.t. $/S$,

$$Z(T) = \bigcup_i Z(T_i)$$

If F was also an alg. stack, this would be the scheme-theoretic image

Definition is tailored to having versal deformation spaces

<u>Thm</u>: If $\xi: \mathcal{X} \to F$ is proper,

$\mathcal{X}$ is an alg. stack l.f.t. $/S$,

$\Delta: F \to F \times F$ is representable by alg. group, l.f.p.,

and F admits versal rings at each f.t. pt., and

Image $\longrightarrow$ Z admits <u>effective</u> versal rings at f.t. pt.s,

then Z is an algebraic stack, l.f.t. $/S$

Can verify effectivity in our case in order to apply theorem.