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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Sean Howe Email/Phone: seanpkh@gmail.com

Speaker's Name: Ben Howard

Talk Title: Cycles on Shimura varieties and applications to Faltings height

Date: 12 / 02 / 2014 Time: 11 : 00 am / pm (circle one)

List 6-12 key words for the talk: Colmez conjecture, Faltings height, Orthogonal Shimura varieties, intersections on Shimura varieties

Please summarize the lecture in 5 or fewer sentences: Introduces Colmez's conjecture on a formula for the Faltings height of a CM type and describes recent work. Focuses on a result showing the averages over the Galois orbit of the Faltings and Colmez heights are equal. Describes the proof using an intersection computation with divisors of special endomorphisms on an integral model of an orthogonal Shimura variety.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
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  - Handouts: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Joint with Andreatta, Goren, Madapusi-Pera

Colmez's conjecture

$$CM \to \begin{array}{c} E \\ |\, 2 \\ E^+ \\ | \\ \mathbb{Q} \end{array} \qquad \Phi = \{e_1, \dots, e_n\} \subseteq \mathrm{Hom}(E, \mathbb{C})$$

s.t. $\mathrm{Hom}(E, \mathbb{C}) = \Phi \sqcup \overline{\Phi}$

Suppose $A/\mathbb{C}$ is an abelian variety with $\mathcal{O}_E \to \mathrm{End}(A)$ and type Φ, i.e. $x \in \mathcal{O}_E$ acts on $\mathrm{Lie}\, A = \mathbb{C}^n$ as

$$\begin{bmatrix} e_1 & & 0 \\ & \ddots & \\ 0 & & e_n \end{bmatrix}$$

Fix a #-field L large enough that A is defined over L and has good reduction $/L$.

$\mathcal{A}/_{\mathrm{Spec}\,\mathcal{O}_L}$ = Néron model.

Line bundle $\det(\pi_* \Omega^1_{\mathcal{A}/\mathcal{O}_L}) \in \mathrm{Pic}(\mathrm{Spec}(\mathcal{O}_L))$

Pick a rational section ω

$$h_\infty^{\mathrm{Falt}}(A, \omega) := \frac{-1}{2[L:\mathbb{Q}]} \sum_{\sigma: L \to \mathbb{C}} \log \left| \int_{A^\sigma(\mathbb{C})} \omega^\sigma \wedge \overline{\omega^\sigma} \right|$$

$$h_f^{\mathrm{Falt}}(A, \omega) := \frac{1}{[L:\mathbb{Q}]} \sum_{\mathfrak{p} \in \mathcal{O}_L} \mathrm{ord}_\mathfrak{p}\, \omega \log N(\mathfrak{p})$$

Faltings height $h^{\mathrm{Falt}}(A) = h_\infty^{\mathrm{Falt}}(A, \omega) + h_f^{\mathrm{Falt}}(A, \omega)$

depends only on $A/\mathbb{C}$.

Thm (Colmez)

$h^{\text{Falt}}(A)$ depends only on (E, Φ)

Call it $h^{\text{Falt}}(E, \Phi)$

$G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\{\text{CM types of } E\}$

$$a_{(E,\Phi)}(\sigma) = |\Phi \cap \sigma(\Phi)|$$

$$a^0_{(E,\Phi)}(\sigma) = \frac{1}{[G_{\mathbb{Q}} : \mathrm{Stab}(\Phi)]} \sum_{\tau \in [G_{\mathbb{Q}} : \mathrm{Stab}(\Phi)]} a_{(E,\tau(\Phi))}(\sigma)$$

Can decompose into Artin characters

$$a^0_{(E,\Phi)} = \sum_{\chi} m(\chi)\chi$$

Artin conductor ↓

Define $$h^{\text{col}}(E, \Phi) = \sum_{\chi} m(\chi)\left[\frac{L'(0,\chi)}{L(0,\chi)} + \frac{1}{2}\log(F_\chi)\right]$$

Conj (Colmez)

$$h^{\text{Falt}}(E,\Phi) = h^{\text{col}}(E,\Phi)$$

Remark In definition of h^{Falt}, can always take L large enough so that ω has no zeroes or poles at finite places (can make line bundle trivial)

- If E is quadratic imaginary, this is Chowla-Selberg
- If $E/\mathbb{Q}$ abelian, proved by Colmez (+ Obus)
- For some non-Galois quartic E, proved by T. Yang (uses Hilbert Modular Surfaces). Can generalize to Shimura varieties with lots of divisors.
- Ongoing work of Bruinier-H-Kudla-Rapoport-Yang - Any E containing a quadratic imag. subfield + some restrictions on CM type

Thm (in progress) (A G H MP)

E any CM field.

$$\sum_{\Phi} h^{\mathrm{Falt}}(E,\Phi) \stackrel{\circ}{=} \sum_{\Phi} h^{\mathrm{col}}(E,\Phi)$$

$\stackrel{\circ}{=}$ means up to $\mathbb{Q}$-linear combination of $\{\log p \mid p \text{ divides } 2\operatorname{disc} E\}$, $\operatorname{disc} E = \Delta$

Let $\Phi_1, \ldots, \Phi_r$ be rep.'s for $G_{\mathbb{Q}}$-orbits of {CM types}.

Each (E, Φ_i) has dual $(E_i^{\#}, \Phi_i^{\#})$

Total reflex algebra $E^{\#} = \prod_i E_i^{\#} \leftarrow \dim 2^n$

$$\Phi^{\#} = \bigsqcup_i \Phi_i^{\#} \subseteq \bigsqcup \operatorname{Hom}(E_i^{\#}, \mathbb{C}) = \operatorname{Hom}(E^{\#}, \mathbb{C})$$

Fact $\quad a^{0}_{(E^{\#}, \Phi^{\#})} = \dfrac{1}{[E:\mathbb{Q}]} \sum_{\Phi} a^{0}_{(E,\Phi)}$

$$\begin{array}{ccc} h^{\mathrm{Falt}}(E^{\#}, \Phi^{\#}) & \stackrel{?}{\stackrel{\circ}{=}} & h^{\mathrm{col}}(E^{\#}, \Phi^{\#}) \\ \| & & \| \\ \dfrac{1}{[E:\mathbb{Q}]} \sum_{\Phi} h^{\mathrm{Falt}}(E,\Phi) & & \dfrac{1}{[E:\mathbb{Q}]} \sum_{\Phi} h^{\mathrm{col}}(E,\Phi) \end{array}$$

Orthogonal Shimura Varieties

Fix $\xi \in (E^{+})^{\times}$ negative at exactly one ∞ place of E^{+}.

$$(V, Q) = (E, \operatorname{Tr}_{E^{+}/\mathbb{Q}}\, \xi x \bar{x})$$

has sign $(2n-2, 2)$

Clifford algebra $\quad C(V) = \left(\bigoplus_{k=0}^{\infty} V^{\otimes k} \right) \Big/ \langle v \otimes v - Q(v) \rangle$

$$1 \to \mathbb{G}_m \to G\mathrm{Spin}(V) \to SO(V) \to 1 \qquad G \subseteq C(V)^{\times}$$

$$D = \{ z \in V_{\mathbb{C}} : [z,z]=0,\ [z,\bar{z}]<0 \} / \mathbb{C}^{\times}$$ Howard (4)

$$\cap$$

$$\mathbb{P}(V_{\mathbb{C}})$$

Shimura data (G, D)

Canonical embedding $E^{\#} \to C(V)$

$$\begin{tikzcd} & & & G \arrow[d] \\ T = \{ x \in E^{\times} \mid x\bar{x} \in \mathbb{Q}^{\times} \} \arrow[r] \arrow[urrr, dashed] & E^{\times} \arrow[r, "\text{Reflex norm}"] & (E^{\#})^{\times} \arrow[r] & C(V)^{\times} \end{tikzcd}$$

$T(\mathbb{R})$ acts on D w/ fixed points $\{z_0^{+}, z_0^{-1}\}$

Morphism $(T, \{z_0^{+}\}) \longrightarrow (G, D)$

$$Y_E(\mathbb{C}) = T_E(\mathbb{Q}) \backslash \{z_0^{+}\} \times T_E(\mathbb{A}_f) / K_E$$

$$\downarrow$$

dim'n $2n-2 \to$ $M(\mathbb{C}) = G(\mathbb{Q}) \backslash D \times G(\mathbb{A}_f) / K$

Canonical models over E.

Kuga - Satake Abelian Scheme

G acts on $C(V)$ by left multiplication.

$G \to GSp(V)$ $\quad M \hookrightarrow$ Siegel moduli.

$$\begin{array}{c} A \\ \downarrow \\ M \end{array}$$ $\dim = 2^{2n-1}$ has right action of $C(V)$

$A|_{Y_E}$ has action of $E^{\#} \otimes C(V) \cong M_{2n}(E^{\#})$

Prop $\exists$ an abelian scheme $\begin{array}{c} B \\ \downarrow \\ Y_E \end{array}$

w/ CM by $\mathcal{O}_{E^\#}$ and type $\Phi^\#$ and a Δ-isogeny
(degree divisible only by primes dividing Δ)

$$A|_{Y_E} \longrightarrow \underbrace{B \times \cdots \times B}_{2^n \text{ times}}$$

Divisors $\mathcal{Z}(m)$ on $\mathcal{M}$

Local system of $\mathbb{Q}$-vs $H_1(A, \mathbb{Q})$ on $M(\mathbb{C})$ defined by $G \to GSp(C(V))$
$\mathbb{V}$ defined by $G \to SO(V)$
$V \hookrightarrow C(V) \xrightarrow{\text{left mult}} \mathrm{End}(C(V))$
inclusion of local systems $\mathbb{V} \to$ ~~[illegible]~~ $\mathrm{End}(H_1(A,\mathbb{Q}))$

Def Given $s \in M(\mathbb{C})$ an endomorphism x of A_s is special if $H_1(x) \in \mathbb{V}_s$

Given connected S, $x \in \mathrm{End}(A_S)$ is special if it is special at one (any) complex point $s \in S$.

$V(A_S) = \{\text{special endomorphisms}\}$
is a positive definite quadratic space via $Q(x) = x \cdot x \in \mathbb{Z}$

$\begin{array}{c} Z(m) \\ {\scriptstyle \text{codim 1 image}} \downarrow \\ M \end{array}$ has S-points $\{x \in V(A_S) : Q(x) = m\}$

Kisin & Vasiu
Integral models over $\mathcal{O}_E[1/\Delta]$: $\mathcal{Z}(m) \to \overset{\mathcal{A}}{\overset{\downarrow}{\mathcal{M}}} \longleftarrow \mathcal{Y}_E$

$\widehat{\mathrm{Pic}}(\mathcal{M}) \to \widehat{\mathrm{Pic}}(\mathcal{Y}_E) \xrightarrow{\widehat{\deg}} \mathbb{R}$ 1-dim'l

$\det(\pi_* \Omega^1_{\mathcal{A}/\mathcal{M}}) \longmapsto \det(\pi_* \Omega^1_{B^{2n}/\mathcal{Y}_E}) \longmapsto h^{\mathrm{Falt}}(E^\#, \Phi^\#)$

Borcherds Products

Fix $F(\tau) = \sum_{m \gg 0} c_F(m) \cdot q^m \in M^!_{2-n}(SL_2(\mathbb{Z}), \mathbb{Z})$

Borcherds constructs a rational section

$$\Psi(F) \text{ of } \det(\pi_* \Omega^1_{A/\mathcal{M}})^{\otimes c_F(0)}$$

(remark: up to some power)

with $\operatorname{div}(\Psi(F)) = \sum_{m>0} c_F(-m) \mathcal{Z}(m)$

Thus, $\underset{\nwarrow}{I}(\operatorname{div}\Psi(F), \mathcal{Y}_E) = \sum_{\sigma: E \to \mathbb{C}} \sum_{y \in \mathcal{Y}_E^\sigma(\mathbb{C})} \log \|\Psi(F)\|$

Intersection mult on $\mathcal{M}$

- Bruinier, Kudla, Yang construct Hilbert modular Eisenstein series

$G_E(\tau, s)$, $\tau \in \mathcal{H} \times \dots \times \mathcal{H}$

$$G_E'(\tau, 0) = \sum_{\substack{\alpha \in E^+ \\ \alpha \gg 0}} a_E(\alpha) \cdot q^\alpha + a_E(0) + \log|\operatorname{Im}(\tau)|$$

Formal q-expansion $\quad G_E(\tau) = a_E(0) + \sum_{\alpha \gg 0} a_E(\alpha) \cdot q^\alpha$

diagonal restriction gives $G(\tau) = \sum_{m \gg 0} a(m) \cdot q^m$

Thm (BKY)

$$-\sum_\sigma \sum_y \log \|\Psi(F)|_y\| = \sum_{m \geq 0} a(m) \cdot c_F(-m)$$

Thm (AGHMP)

$I(\mathcal{Z}(m), \mathcal{Y}_E) = -a(m)$,

so $\quad I(\operatorname{div}\Psi(F), \mathcal{Y}_E) = -\sum_{m>0} a(m)\, c_F(-m)$

so $\quad c_F(0) \cdot h^{\text{Falt}}(E^\#, \Phi^\#) = a(0) \cdot c_F(0) \quad$ (by adding)

$$h^{\text{Falt}} = a(0) = \frac{L'(0, \chi_{E/E^+})}{L(0, \chi_{E/E^+})} = h^{\text{Col}}(E^\#, \Phi^\#)$$