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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Sean Howe Email/Phone: seanpkh@gmail.com

Speaker's Name: Ana Caraiani

Talk Title: On the Hodge-Tate period map for Shimura varieties of Hodge type

Date: 12 / 2 / 2014 Time: 2 : 00 am / pm (circle one)

List 6-12 key words for the talk: Hodge-Tate period map, Perfectoid Shimura Varieties, Relative p-adic comparison theorem

Please summarize the lecture in 5 or fewer sentences: Explains the construction of Hodge-Tate period maps at infinite level for Shimura varieties of Hodge type, and the construction of automorphic vector bundles via pullback along the period map. Sketches the proof which hinges upon comparing the Hodge-Tate and Hodge-de Rham filtrations using relative comparison theorems.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Joint work w/ Peter Scholze

Applications ~ Already p-adic aut. forms, Galois rep's attached to cohomology classes, etc.

Hodge theory: $X/_E$ projective smooth variety $/E$

$E \hookrightarrow \mathbb{C}$, X^{an} = complex analytic vector space

Betti cohomology $H^*(X^{an}, \mathbb{Q}) \otimes \mathbb{C} = \bigoplus_{p,q} H^q(X^{an}, \Omega^p_{X^{an}})$

holomorphic p-forms

Hodge decomposition.

de Rham cohomology $H^*_{dR}(X)/E \simeq \mathbb{H}(X, \Omega^\circ_X)$

algebraic de Rham complex

Hodge-deRham filtration on $H^*_{dR}(X)$.

Comparison isomorphism $H^*(X^{an}, \mathbb{Q}) \otimes \mathbb{C} \simeq H^*_{dR}(X) \otimes_E \mathbb{C}$

Hodge filtration $\mathrm{Fil}^p H^n = \bigoplus_{p' \geq p} H^{p',q'}$

$H^*_{dR}(X)$ exists also for adic spaces $/ \mathrm{Spa}(K, \mathcal{O}_K)$

for $K/\mathbb{Q}_p$ finite.

$A/_{\mathbb{C}}$ an abelian variety $\Longleftrightarrow$ HS of type $(-1,0)$ $(0,-1)$

$$0 \to H^0(A, \Omega^1_A) \to H^1_{dR}(A) \to H^1(A, \mathcal{O}_A) \to 0$$

Hodge-de Rham filtration.

Shimura varieties

(G, X), $G/\mathbb{Q}$ reductive group, X = conj. class of $h: \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$

Tannakian group for real Hodge structures

satisfying certain axioms

$X \simeq G(\mathbb{R})/K^\circ_\infty A^\circ_\infty$ Herm. symm. domain

Assume (G, X) is of Hodge type, i.e. $\exists$ an embedding of Shimura data $(G,X) \hookrightarrow (\widetilde{G}, \widetilde{X})$

$$\widetilde{G} = GSp(V, \psi)$$

ψ symplectic form

The Borel embedding:

h determines Hodge cochar $\mu : \mathbb{G}_m \to G_{\mathbb{C}}$

$\mathbb{G}_m$: Tannakian group for graded vector spaces

$\rightsquigarrow$ Filtration on $\mathrm{Rep}_{\mathbb{C}}(G)$ (descending)

h is a real HS $\Rightarrow$ Fil_μ is Hodge-deRham filtration

$$P_\mu^{std} = \{ g \in G \mid \lim_{t \to 0} \mathrm{ad}(\mu(t)) g \text{ exists} \}$$

$$= \text{stabilizer in } G \text{ of } \mathrm{Fil}_\mu$$

$$P_\mu = \{ g \in G \mid \lim_{t \to \infty} \mathrm{ad}(\mu(t)) g \text{ exists} \}$$

$$= \text{stabilizer of opposite (complex conj.) filtration } \mathrm{Fil}^\mu \text{ on } \mathrm{Rep}_{\mathbb{C}} G.$$

$$G(\mathbb{C}) / P_\mu^{std}(\mathbb{C}) = \mathcal{F}\ell_G^{std} = \text{moduli of filtrations conj. to } \mathrm{Fil}_\mu$$

$$G(\mathbb{C}) / P_\mu(\mathbb{C}) = \mathcal{F}\ell_G = \text{`` '' } \mathrm{Fil}^\mu$$

Borel embedding $X \hookrightarrow \mathcal{F}\ell_G^{std}$ $\quad (\mathcal{H} \hookrightarrow \mathbb{P}^1(\mathbb{C}))$

$h \longmapsto \mu_h$

holomorphic (Also $X \hookrightarrow \mathcal{F}\ell_G$ antiholomorphic)

$K \subseteq G(\mathbb{A}_F)$ compact open

$\hookrightarrow \quad Sh_K(G,X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_F)/K$

X universal cover of each connected component.

Sh_K, Fl_G^{std} have models over E = reflex field = field of def'n of μ

To define automorphic vector bundles / Sh_K:

$$\operatorname{Rep} P_\mu^{std} \simeq \{ G(\mathbb{C})\text{-equivariant vector bundles on } Fl_G^{std} \}$$

$$\to \{ G(\mathbb{R})\text{-equivariant vector bundles on } X \}$$

Algebraic, defined over E (Harris & Milne)

descend to aut. v.b. on Sh_K

Idea: Consider $(G,X) \hookrightarrow (\widetilde{G}, \widetilde{X})$ a Siegel embedding

$\exists \ (s_\alpha) \in V^{\otimes}$, finitely many s.t. $G = \operatorname{Stab}(s_\alpha)$

$Sh_K \hookrightarrow \widetilde{Sh}_{\widetilde{K}} \times_{\mathbb{Q}} E$ closed embedding

Restriction $\pi: A \to Sh_K$ gives an AV.

$$\mathcal{V}_{dR} := R^1 \pi_{dR *} \mathcal{O}_A$$

$$\mathcal{G}_{dR} = G\text{-torsor} / Sh_K$$
$$= \{ \beta : \mathcal{V}_{dR} \to V_\mu \otimes \mathcal{O} \}$$
$$\beta(\underbrace{s_{\alpha,dR}}) = s_\alpha$$

de Rham realization of s_α (Horizontal sections of $\mathcal{V}_{dR}^{\otimes}$)

$$\begin{tikzcd} & \mathcal{G}_{dR} \arrow[dl, "k"'] \arrow[dr, "j"] & \\ Sh_K & & Fl_G^{std} \end{tikzcd}$$

$\mathcal{G}_{dR}$ has a P_μ^{std}-structure (obtained from P_μ^{std}-torsor $\mathcal{P}_{dR}$ via $P_\mu^{std} \to G$)

$$\mathcal{P}_{dR} = \{ \beta \in \mathcal{G}_{dR} \mid \beta(\operatorname{Fil}^{\circ} V_{dR}) \}$$

= Hodge deRham Filt.

Get automorphic vector bundles on Sh_K by pulling back via j and then descending via k

p-adic picture:

$C = \hat{\bar{\mathbb{Q}}}_p$, $\varprojlim_{K_p} (Sh_{K^pK_p} \times_E C)^{ad} \sim \mathcal{S}_{K^p}$,

Perfectoid Shimura variety.

There is a map of adic spaces

$$\pi'_{HT} : \mathcal{S}_{K^p} \to \mathcal{F}l^{ad}_{\tilde{G}} \circlearrowleft \tilde{G}(\mathbb{Q}_p)$$

$G(\mathbb{Q}_p) \circlearrowleft \mathcal{S}_{K^p}$, $\mathcal{F}l^{ad}_{\tilde{G}} = (\mathcal{F}l_{\tilde{G}} \times C)^{ad}$

$((G, X) \hookrightarrow (\tilde{G}, \tilde{X}))$

π'_{HT} is equivariant for the $G(\mathbb{Q}_p)$-action,
and (varying K^p in a tower) for the $G(\mathbb{A}_f^p)$-action
(trivial action on $\mathcal{F}l_{\tilde{G}}$)

If A/C is an abelian variety of dim g,

$A/C \rightsquigarrow$ pt of $\tilde{\mathcal{S}}_{\tilde{K}^p}$

$H^1_{ét}(A, \mathbb{Q}_p) \xrightarrow{\sim} \mathbb{Q}_p^{2g}$

Hodge-Tate filtration

$0 \to H^1(A, \mathcal{O}_A) \to H^1_{ét}(A, \mathbb{Q}_p) \otimes C \to H^0(A, \Omega^1_A)(-1) \to 0$

gives point in $\mathcal{F}l_{\tilde{G}}$

Theorem (Scholze, C.) 1) $\exists$ $G(\mathbb{Q}_p)$-eq. map of adic spaces

$$\pi_{HT} : \mathcal{S}_{K^p} \to \mathcal{F}l^{ad}_G = (\mathcal{F}l_G \times_E C)^{ad}$$

Compatible with π'_{HT}

2) There is an isomorphism of tensor functors given by

$M_\mu = \mathrm{Cent}(\mu)$
Common Levi of P_μ, P_μ^{std}

$$\begin{array}{ccc} \mathrm{Rep}\, M_\mu & \longrightarrow & \{ G(\mathbb{Q}_p)\text{-eq. vector bundles on } \mathcal{F}l^{ad}_G \} \\ \downarrow & & \downarrow \pi^*_{HT} \\ \{ \text{Aut. VB on } \mathcal{S}_{K^p} \} & \longrightarrow & \{ G(\mathbb{Q}_p)\text{-eq. vector bundles on } \mathcal{S}_{K^p} \} \end{array}$$

Can also write in K^p tower with $G(\mathbb{A}_f)$-action, trivial on $\mathcal{F}l^{ad}_G$

Ideas: 1) p-adic de Rham comparison in Families (Scholze) induces HT Filtration

2) $\mathcal{V}_p = R^1\pi_* \mathbb{Q}_p \qquad \pi: \mathcal{A} \to \mathcal{S}_K$

Construct M_μ, P_μ, G-torsors trivializing $\mathcal{V}_p \otimes \hat{\mathcal{O}}_{\mathcal{S}_K}$

1) Comparison: $\mathcal{V}_p \otimes \hat{\mathcal{O}}_{\mathcal{S}_K} \xrightarrow{\sim} R^1\pi_* \hat{\mathcal{O}}_{\mathcal{A}}$

(X adic space $\rightsquigarrow$ $\hat{\mathcal{O}}_X$ sheaf on $(X)_{\text{proét}}$)

First step in relative Hodge-Tate Filtration

$$0 \to R^1\pi_* \mathcal{O}_{\mathcal{A}} \otimes_{\mathcal{O}_{\mathcal{S}_K}} \hat{\mathcal{O}}_{\mathcal{S}_K} \to R^1\pi_* \hat{\mathcal{O}}_A$$

Another way: p-adic - deRham comparison

Filtration $\to$ $\mathbb{B}^{(+)}_{dR,\mathcal{A}}$ relative period sheaf on $(\mathcal{A})_{\text{proét}}$

$Gr^0 \mathbb{B}^{(+)}_{dR,\mathcal{A}} = \hat{\mathcal{O}}_{\mathcal{A}}$

$$" \left(W(\mathcal{O}^+_{\mathcal{A}^\flat})[\tfrac{1}{p}]\right)^{\wedge}_{\ker\theta}$$

$$\theta: W(\mathcal{O}^+_{\mathcal{A}^\flat}) \to \mathcal{O}^+_{\mathcal{A}}$$

$\mathcal{O}\mathbb{B}^{(+)}_{dR}$ structural period sheaf

$$R^1\pi_* \mathbb{B}^{(+)}_{dR,A} \otimes_{\mathbb{B}^+_{dR,\mathcal{S}_K}} \mathbb{B}_{dR,\mathcal{S}_K} \simeq R^1\pi_{dR*} \mathcal{O}_{\mathcal{A}} \otimes_{\mathcal{O}_{\mathcal{S}_K}} \mathbb{B}_{dR,\mathcal{S}_K}$$

compatible w/Filtration and connection

$$R^1\pi_* \hat{\mathcal{O}}_A = Gr^0(LHS)^{\nabla=0}$$

Hodge-Tate filtration $\to M = R^1\pi_* \mathbb{B}^{(+)}_{dR,\mathcal{A}}$ loc system on $\mathcal{S}_K$ (corresponding to étale cohom)

$\to M_0 = (R^1\pi_{dR*} \mathcal{O}_{\mathcal{A}} \otimes_{\mathcal{O}_{\mathcal{A}}} \mathcal{O}\mathbb{B}^{(+)}_{dR})^{\nabla=0}$ loc system corresponding to the deRham cohomology

Hodge deRham Filtration

Just like Scholze's talk, two $\mathbb{B}^+_{dR,\mathcal{F}_K}$ local systems sitting inside same $\mathbb{B}_{dR,\mathcal{F}_K}$ local system.
(on points, gives 2 lattices)

This gives you a Hodge de Rham filt on M_0, Hodge-Tate on M,
related by comparison isomorphism

$\mathbb{B}^+_{dR}$ - structure on M_0
induces H-T Filtration on
$R^1\pi_* \hat{\mathcal{O}}_A = Gr^0(LHS)^{\nabla=0}$

and vice versa to get Hodge-dRham Fill.

For part 2) check that tensors preserve HT Filtration
using this comparison + preservation of Hodge Filtration