MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Sean Howe **Email/Phone:** seanpkh@gmail.com

Speaker's Name: Raghuram

Talk Title: Special Values of automorphic L-functions and congruences

Date: 12 / 03 / 2014 **Time:** 9 : 00 am / pm (circle one)

List 6-12 key words for the talk: Congruences of automorphic forms, special values of adjoint L functions, automorphic forms on GL_n

Please summarize the lecture in 5 or fewer sentences: Proves that outside a finite set of primes, divisibility of a special value of an adjoint L function of a cohomological cuspidal automorphic representation of $GL_n(F)$ by a prime implies the existence of a congruence mod that prime with another cohomological cuspidal automorphic rep, generalizing a classical theorem of Hida on modular forms for $GL_2(\mathbb{Q})$.

CHECK LIST

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Theorem (Hida '81) Raghuram ①

$F \in S_K(N)_{Prim}$ / $p \mid \underset{\text{"periods"}}{L(1, Ad^0, F)} \Rightarrow p$ is a congruence prime, i.e. $\exists\ g \in S_K(N),\ g \neq F$ s.t. $a_n(F) \equiv a_n(g)\ (p)$

Eknath Ghate, Eric Urban $GL(2)$/ Im. quadratic

Mladev Dimitrov, Hida + Namikawa $GL(2)$ / # field.

Joint w/ Baskar ~~Bala~~ ~~Balasubramanyan~~ Balasubramanyam

$GL(n)$ / # field

§1 Analytic theory of L-Functions for $GL(n) \times GL(m)$

Let (π, V_π) be a cuspidal automorphic representation of $GL(n)/F$, F # Field

Let $(\tilde{\pi}, V_{\tilde{\pi}})$ - -

$e \in V_\pi$, $\tilde{e} \in V_{\tilde{\pi}}$, $\underset{\text{Schwartz}}{\Phi \in \mathcal{S}(\mathbb{A}_F^n)}$ $\eta = \omega_\pi \cdot \omega_{\tilde{\pi}}$

$$D(s, e, \tilde{e}, \Phi) = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} e(g)\,\tilde{e}(g)\,\underset{\text{Eisenstein series}}{E(s, g, \Phi, \eta)}\,dg \quad \forall s$$

$e \Leftrightarrow \underset{\text{Whittaker vector (assume pure tensor)}}{W_e = \otimes w_v} \in W(\pi, \underset{\text{additive char}}{\psi})$ $\tilde{e} \Leftrightarrow W_{\tilde{e}} = \otimes_v \tilde{w}_v \in W(\tilde{\pi}, \psi^{-1})$

$$\ldots \text{unfolding} = \prod_v \int_{N(F_v)\backslash G(F_v)} w_v(g_v)\,\tilde{w}_v(g_v)\,\Phi(\underset{\text{last row of } g_v}{e_n g_v})\,|\det g_v|^{s-1/2}\,dg_v$$

Ψ_v is this integral

$$= \underset{(L^S \text{ From } v \text{ outside } S)}{L^*(s, \pi \times \tilde{\pi})} \prod_{v \in S} \frac{\Psi_v(s, w_v, \tilde{w}_v, \Phi_v)}{L(s, \pi_v \times \tilde{\pi}_v)}$$

Take $\tilde{\pi}$ = contragredient, get

$$= \zeta_F(s) \cdot L(s, \mathrm{Ad}^0, \pi) \prod_{v \in S} c_v(s)$$

$$\operatorname*{Res}_{s=1} D(s, e, \tilde{e}, \Phi) = (\ldots) \langle e, \tilde{e} \rangle_{\mathrm{Pet}}$$
$$= (\ldots) L(1, \mathrm{Ad}^0, \pi)$$

$v \in S \setminus S_\infty$, $c_v(s)|_{s=1}$ is nice

Löic Grenié's thesis

$v \in S_\infty$, "cohomological vectors", Binyong Sun, $c_v(1) \neq 0$

<u>Prop</u> $\exists$ cusp forms $e \in V_\pi$, $\tilde{e} \in V_{\tilde{\pi}}$ s.t.

$$\langle e, \tilde{e} \rangle = \frac{L(1, \mathrm{Ad}^0, \pi)}{\Omega_F \, P_{\mathrm{ram}}(\pi) \, P_\infty(\pi)}$$

Now: Interpret this purely analytic identity in cohomology.

§ 2 Cohomology of arithmetic groups

$G = R_{F/\mathbb{Q}}(GL_n/F) \supseteq B \supseteq T \supseteq Z \supseteq S = \mathbb{G}_m/\mathbb{Q}$

$\lambda \in X^*_+(T \times E)$, $\lambda = (\lambda^\tau)_{\tau: F \to E}$ strongly pure

dominant integral

$F \hookrightarrow E$, E / $\mathbb{Q}$ Galois

$M_{\lambda, E}$ = algebraic irrep. of G/E

$K_f \subseteq G(\mathbb{A}_f)$ open compact

$K_\infty = \prod_{v \in S_r} O(n) \cdot \prod_{v \in S_c} U(n) \cdot S(\mathbb{R})$, $\pi_0(K_\infty) = \prod_{v \in S_r} \mathbb{Z}/2$

$S^G_{K_F} = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K^0_\infty K_F$, $\widetilde{\mathcal{M}_{\lambda,E}}$ a sheaf on $S^G_{K_F}$

$$H^\bullet(S^G_{K_F}, \widetilde{\mathcal{M}_{\lambda,E}}) \supseteq H^\bullet_!(S^G_{K_F}, \widetilde{\mathcal{M}_{\lambda,E}})$$

$E = \mathbb{C}$, $\quad H^\bullet(\ldots) \supseteq H^\bullet_!(\ldots) \supseteq H^\bullet_{cusp}(S^G_{K_F}, \widetilde{\mathcal{M}_{\lambda,\mathbb{C}}})$

$$H^\bullet_! = \mathrm{Im}(H^\bullet_c \to H^\bullet)$$

$\mathcal{H}^G_{K_F} \times \pi_0(G(\mathbb{R}))$ acts on all.

<u>Facts</u> 1) $H^\bullet_{cusp}(\ldots) \neq 0 \iff b^F_n \leq \cdot \leq t^F_n$

$$b^F_n = [\tfrac{n^2}{4}] \# S_r + \tfrac{(n)(n-1)}{2} \# S_c$$
$$b^F_n + t^F_n = \dim(S^G_{K_F})$$

2) $\cdot \in \{b^F_n, t^F_n\}$ (extremal degrees by 1))

$$H^\bullet_{cusp}(S^n_{K_F}, \widetilde{\mathcal{M}_{\lambda,\mathbb{C}}}) = \begin{cases} \bigoplus_{\pi \in Coh(G,K_F,\lambda)} \bigoplus_{\varepsilon \in \pi_0(G(\mathbb{R}))} \pi_F^{K_F} \otimes \varepsilon, & n \text{ even} \\ \bigoplus_{\pi \in Coh(G,K_F,\lambda)} \pi_F^{K_F} \otimes \varepsilon_{\pi_\infty}, & n \text{ odd} \end{cases}$$

<u>Betti-Whittaker periods</u> (Hida, Harder, Mahnkopf, R-Shahidi, R., Grobner-Harris, ~~[illegible]~~ (-Lapid))

$\pi \in Coh(G, K_F, \lambda)$

$$W(\pi_F) \xrightarrow[\sim]{F_\varepsilon} H^\bullet(\pi_F)$$

whittaker model (Some type of Eichler-Shimura iso)

$$\begin{array}{ccc} W_{/\mathbb{C}} & \xrightarrow[\sim]{F} & V_{/\mathbb{C}} \\ \uparrow & & \uparrow \\ W^0_{/E} & & V^0_{/E} \end{array}$$

$\cdot = b^F_n \quad p^\varepsilon(\pi) \longleftrightarrow$ "Deligne periods"

$\cdot = t^F_n \quad q^\varepsilon(\pi) \longleftrightarrow$ "Beillinson periods"

$(b \neq t)$

$\exists\ P(F)$

s.t. $F(W^0) = P(F) V$

(uniqueness of rational structures up to homothety)

$$\begin{tikzcd} H^b_! (\pi_F \times \varepsilon) \times H^t_! (\tilde{\pi}_F \times \tilde{\varepsilon}) \arrow[r, "\text{Poincaré}"] & \mathbb{C} \arrow[d, equal] \\ W(\pi_F) \times W(\tilde{\pi}_F \times \varepsilon) \arrow[u] \arrow[d] & \\ V_\pi \times V_{\tilde{\pi}} \arrow[r, "\text{Petersson}"] & \mathbb{C} \end{tikzcd}$$

$\tilde{\varepsilon} = (-1)^{n-1}\varepsilon$

<u>Theorem</u> Let $\pi \in \mathrm{Coh}(\pi, K_F, \lambda)$ ε-permissible for π (In sense that it appears in previous statement For n odd)

Then

$$\sigma\left(\frac{L(1, Ad^0, \pi)}{\Omega_F \, P_{ram}(\pi) P_\infty(\pi) \, p^\varepsilon(\pi) \, q^{\tilde{\varepsilon}}(\tilde{\pi})} \right) = \frac{L(1, Ad^0, {}^\sigma\pi)}{\Omega_F \, P_{ram}({}^\sigma\pi) \ldots}$$

$$\Rightarrow \underbrace{\frac{L(1, Ad^0, \pi)}{\text{"Periods"}}}_{=: L^{alg}(1, Ad^0, \pi, \varepsilon)} \in \mathbb{Q}(\pi) \text{ — \# field}$$

If F is CM, $\pi^\vee = \overline{\pi}$, then this is also proved by Grobner–Harris–Lapid

$$\begin{tikzcd} \mathbb{C} \arrow[r, phantom, "\cong"] \arrow[d, dash] & \overline{\mathbb{Q}_p} \arrow[d, dash] & & \\ F, \mathbb{Q}(\pi), \mathbb{Q}(\lambda) \arrow[r, hook] \arrow[d, dash] & E \arrow[r, phantom, "\supset"] \arrow[d, dash] & \mathcal{O}_E \arrow[r, phantom, "\supset"] & \mathfrak{p}_E \\ \mathbb{Q} \arrow[r, dash] & \mathbb{Q}_p & & \end{tikzcd}$$

$$\begin{array}{c} W(\pi_f) \longrightarrow H^0(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathbb{C}})(\pi_f \times \varepsilon) \\ \uparrow \\ H^0(--, \widetilde{\mathcal{M}_{\lambda,E}}) \\ \uparrow \\ \overline{H^0}(--, \mathcal{M}_{\lambda,\mathcal{O}_E}) \end{array}$$

means image in rational cohomology

<u>Theorem</u> (Baskar + R)

$\pi \in \mathrm{Coh}(G, K_f, \lambda)$, ε-permissible sign

(i) $\exists$ Finite set S of prime s.t. if $p \notin S$,

$v_p(L^{alg}(1, Ad^0, \pi, \varepsilon)) > 0 \Rightarrow \exists\ \pi' \in \mathrm{Coh}_!(G, K_f, \lambda)$

s.t. $\pi' \neq \pi$,

$\pi' = \pi \bmod \mathfrak{p}_E$

(i.e. Hecke eigenvalues for π, π' congruent almost everywhere)

(λ regular $\Rightarrow H^0_! = H^0_{cusp}$)

(ii) Suppose λ is a parallel weight, then $\exists$ Finite set S_2 s.t.

$v_p(L^{alg}(1, Ad^0, \pi, \varepsilon)) > 0 \Rightarrow \exists\ \pi' \in \mathrm{Coh}_!(G, K_f, \lambda)$

$\pi' \not\cong \pi$

and $\pi' \cong \pi \bmod \mathfrak{p}_E$

<u>"Proof"</u> $V/_E = V_1 \oplus V_2/_E$ $\quad \widetilde{V}/_E \quad$ $V \times \widetilde{V} \to E$

$\qquad L/_{\mathcal{O}} \supseteq L_1 \oplus L_2 \quad \widetilde{L}/_{\mathcal{O}} \supseteq \widetilde{L}_1 \oplus \widetilde{L}_2 \quad L \times \widetilde{L} \to \mathcal{O}$

want perfect pairing

$\mathcal{C}(L; V_1, V_2) = L \big/ (L \cap V_1) \oplus (L \cap V_2)$ congruence module

$\mathfrak{p} \mid L^{alg}(1, Ad^0, \tilde{\pi}, \varepsilon) \Longrightarrow \mathfrak{p} \mid \mathrm{disc}(L_1 \times \tilde{L}_1)$

$\Longleftarrow$ hard

$\Longleftrightarrow \mathfrak{p} \in \mathrm{Supp}\, \mathcal{C}(L; V_1, V_2)$

$V = H^b_!(S^G_{K_F}, M_{\lambda, E})$ $\quad \tilde{V} = H^t(S^G_{K_F}, \tilde{M}_{\lambda, E})$

$V_1 = \bigoplus_{\varepsilon} H^b_!(\ldots)(\pi_F \times \varepsilon)$ $\quad V_2 =$ "Everything else"

$L = \bar{H}^b_!(S^G_{K_F}, M_{\lambda, \mathcal{O}})$ ... get a proof of (i) of theorem.