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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Sean Howe **Email/Phone:** seanpkh@gmail.com

Speaker's Name: Akshay Venkatesh

Talk Title: The Exterior algebra in the cohomology of an arithmetic group

Date: 12 / 03 / 2014 **Time:** 10 : 30 am / pm (circle one)

List 6-12 key words for the talk: Cohomology of arithmetic groups, derived Hecke-algebra, Taylor-Wiles method

Please summarize the lecture in 5 or fewer sentences: Points out a phenomenon by which Hecke eigensystems appear with multiplicities in different degrees of cohomology with dimensions like an exterior algebra, then develops an explanation. Gives a hypothetical motivic exterior algebra that should act rationally, then gives concrete constructions in the l-adic and complex realizations. The construction in the l-adic case uses the action of the derived Hecke algebra together with the Taylor-Wiles method.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

$\Gamma = PGL_n(\mathbb{Z})$ $\quad H^*(\Gamma, \mathbb{C}) \circlearrowleft$ Hecke

Issue Same Hecke eigensystem appears in multiple degrees.

Goal "Explain" by extra action on H^*

π: Cusp form for PGL

$H^*(\Gamma, \mathbb{C})_\pi \sim$ same Hecke eigenvalues as π.

$$\delta = \left[\tfrac{n-1}{2}\right]$$

$$\dim H^j(\Gamma, \mathbb{C})_\pi = (0, \dots, 0, 1, \delta, \tbinom{\delta}{2}, \dots, \delta, 1, 0, \dots, 0) \quad (*)$$

(if non-zero)

So, we'd like to make an action of $\Lambda^* \mathbb{C}^\delta$ on H^*

General setup:

G semisimple, $\Gamma \leq G$, π a cuspidal tempered rep. of G (remark – tempered shouldn't be necessary)

$H^*(\Gamma, \mathbb{C})_\pi$

Take Hecke e.v. $/\mathbb{Q}$, look at $H^*(\Gamma, \mathbb{Q})_\pi$

Dimensions are like $(*)$

$\delta = \operatorname{rank}(G(\mathbb{R})) - \operatorname{rank}(K_\infty)$ – can be 0, e.g. Shimura variety.

Goal Make δ-dim'l $\mathbb{Q}$-v.s. L_π and action of $\Lambda^* L_\pi$ on $H^*(\Gamma, \mathbb{Q})_\pi$

Status Can define L_π and actions of

$$\Lambda^*(L_\pi \otimes \mathbb{Q}_p \text{ or } \mathbb{C}) \circlearrowright H^*(\Gamma, \mathbb{Q}_p \text{ or } \mathbb{C})_\pi$$

Conjecture $\Lambda^* L_\pi$ preserves $H^*(\Gamma, \mathbb{Q})$

Prasanna + V. have evidence for $\otimes \mathbb{C}$ version.

Remarks

- For Shimura varieties, repeated eigensystems in different degrees explained by Lefschetz
- corresponding story of derived deformation ring where exterior algebra arises naturally
- Work in progress

§2 Def'n for GL_n

$\pi \to n$ dim'l motive $M/\mathbb{Q} \to \rho_\ell : G_\mathbb{Q} \to GL_n(\mathbb{Q}_\ell)$

$$L_\pi^* = \mathrm{Ext}^1_{M\text{-}mot}(M, M(1)) \xrightarrow[(\otimes \mathbb{Q}_\ell)]{\text{conj}} \mathrm{Ext}^1_F(\rho_\ell, \rho_\ell(1))$$

$$= \mathrm{Ext}^1_{mm}(\mathbb{Q}, M^\vee \otimes M(1)) \qquad \qquad \| $$

$$H^1_F(\check{\rho}_\ell \otimes \rho_\ell(1))$$

How to think of L_π

a) L_π shows up in Beillinson conj. for $L(1, Ad, \pi)$

b) $L_\pi \otimes \mathbb{Q}_p$ measure of obstructedness of deforming ρ_p.

General G

$\pi \xrightarrow{?}$ Family of motives indexed by $\mathrm{Rep}({}^LG)$
N = "adjoint" motive
$\dim N = \dim G$

$$\mathrm{Ext}^1(\mathbb{Q}, N(1))$$

§3A $G_\ell = G(\mathbb{Q}_\ell) \supseteq K_\ell = \underline{G}(\mathbb{Z}_\ell)$

k ring, $\frac{1}{\ell} \in k$

$$HA_k = k[K_\ell \backslash G_\ell / K_\ell]$$
$$= \mathrm{Hom}_{G_\ell}(k[G_\ell/K_\ell], k[G_\ell/K_\ell]),$$

$$dHA_k = \bigoplus_{i=0} \mathrm{Ext}^i_{G_\ell}(-,-)$$
graded algebra.

$HA_k \curvearrowright H^*(\Gamma, k)$,

but also $dHA_k \curvearrowright \ldots$
(but doesn't preserve degree — graded action)

$$dHA_k = \bigoplus_{x \in K_\ell \backslash G_\ell / K_\ell} H^*(K_x, k)$$
$$K_x = K_\ell \cap x K_\ell x^{-1}$$

<u>Example</u> $G = PGL_2(\mathbb{Q}_\ell)$, $K_\ell = PGL_2(\mathbb{Z}_\ell)$

$x = \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}$, $K_x = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_\ell) \mid \ell \mid c \right\}$

$$K_x \longrightarrow \underset{[d]}{(\mathbb{Z}/\ell)^\times}$$

$H^i(K_x, k)$ vanishes if $\ell - 1$ invertible in k

In fact, the most useful case is $dHA_{\mathbb{Z}/p^n}$ when $\ell \equiv 1 \bmod p^m$

How does $\alpha \in H^1(K_x, k)$ act on $H^*(\Gamma, k)$

<u>usual</u>

$$\begin{tikzcd} & \Gamma_0(\ell) \arrow[dl, "\pi_1^*"'] \arrow[dr, "\pi_2^*"] & \\ \Gamma & & \Gamma \end{tikzcd}$$

<u>derived</u>

$$\begin{tikzcd} \Gamma_0(\ell) \arrow[r, "\cup\alpha"] \arrow[d, "\pi_1^*"'] & \Gamma_0(\ell) \arrow[d, "\pi_{2*}"] \\ \Gamma & \Gamma \end{tikzcd}$$

$\Gamma \to PGL_2(\mathbb{Z}_\ell)$
$\Gamma_0(\ell) \to K_x$

Get $\quad$ d. Hecke $\circlearrowright H^*(\Gamma, \mathbb{Z}/p^n)$ $\quad$ (generated by all end from d. Hecke for all ℓ at once)

$$\rightsquigarrow$$

d. Hecke $\circlearrowright H^*(\Gamma, \mathbb{Z}_p)_\pi$

Using Taylor-Wiles method (in Calegari-Geraghty) (... assumptions)

d. Hecke $\cong \Lambda^* V$ $\quad$ $V \cong L_\pi \otimes \mathbb{Q}_p$

δ-dim'l

§3B Relation w/ T-W, action of $L_\pi \otimes \mathbb{Q}_p$

Remark Top group $G \circlearrowright X$, k

a) $H_*^{top}(G, k)$ ~~acts~~ $\circlearrowright H_*(X, k)$

b) $H^*_{gp}(G, X) \circlearrowright H^*(X/G, k)$

IF $G =$ torus, (a) and (b) are related by Koszul duality

If $G =$ p-group, $k = \mathbb{Z}/p\mathbb{Z}$, "morally true"

For us, $X = \Gamma_1(\ell) \backslash$ Symm. space

$G = (\mathbb{Z}/\ell)^* \quad (\cong \Gamma_0(\ell)/\Gamma_1(\ell))$

$\boxed{R_\infty \quad S_\infty}$

$R_\infty \otimes_{S_\infty} \mathbb{Z}_p$

$\mathrm{Ext}_{S_\infty}(\mathbb{Z}_p, \mathbb{Z}_p)$

a) $k[G] \circlearrowright H_*(\Gamma_1(\ell))$ — TW-method

b) $H^*(G)$ acts on $H^*(\Gamma_0(\ell))$ (by cup product)

How to "index" by L_π.

Take $\ell \equiv 1\ (p^n)$ TW-prime.

$$\underbrace{H^1(\text{torus}(\mathbb{F}_\ell), \mathbb{Z}/p^n)}_{\text{lives in dH}\Lambda} \times H^1_F(G_{\mathbb{Q}}, \mathrm{Ad}\rho(1)) \to \mathbb{Z}/p^n$$

$$\downarrow$$

$$H^1_F(G_{\mathbb{Q}_\ell}, \mathrm{Ad}\rho(1))$$

$$\downarrow$$

$$\text{Torus}(\mathbb{F}_\ell)$$

$$\Lambda^*(L_\pi \otimes \mathbb{Q}_p) \hookrightarrow H^*(\Gamma, \mathbb{Q}_p)_\pi$$

Conj. $\Lambda^* L_\pi$ preserves $H^*(\Gamma, \mathbb{Q})$

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§4 $L_\pi \otimes \mathbb{C}$ (joint w/ Prasanna)

$$\Lambda^*(L_\pi \otimes \mathbb{C}) \hookrightarrow H^*(\Gamma, \mathbb{C})_\pi$$
$$= H^*((\mathfrak{g}, K), \pi_\infty)$$
$$\| $$
$$H^{min}(\ldots) \otimes \Lambda^* \mathfrak{a}^*$$

$\mathfrak{a}$ = Lie (split part of fund. Cartan)

$\dim \mathfrak{a} = \delta$.

$$L_\pi \otimes \mathbb{C} \xrightarrow[\text{regulator}]{} \mathfrak{a}^*$$

Conj $\Lambda^* L_\pi$ preserves $H^*(\Gamma, \mathbb{Q})_\pi$

This conjecture predicts period matrix of $H^j(\Gamma, \mathbb{C})_\pi$ for each j
(e.g. if $\dim H^j = 1$, predicts $\int_{\partial CH_2(\mathbb{Q})} \omega$ $\langle \omega, \bar{\omega} \rangle = 1$ up to $\mathbb{Q}^*$)

Prasanna + V have verified predictions in some cases
(compatible w/ Beillinson's conj.)
via automorphic periods and analytic torsion.