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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Sean Howe Email/Phone: seanpkh@gmail.com

Speaker's Name: Dipendra Prasad

Talk Title: Branching laws and period integrals for non-tempered representations

Date: 12/03/2014 Time: 12:00 am / pm (circle one)

List 6-12 key words for the talk: Branching laws, non-tempered representations, classical groups

Please summarize the lecture in 5 or fewer sentences: Explains some branching laws for restriction of non-tempered automorphic representations for classical groups, with many explicit examples and applications.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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Branching laws for non-tempered representations

Dipendra Prasad
Tata Institute of Fundamental Research

Automorphic forms, Shimura viarieties, Galois representations, and L-functions
MSRI

December 03, 2014

(joint work with Wee Teck Gan and B. Gross)

Introduction

Branching laws for compact unitary groups (from $U(n+1)$ to $U(n)$):

$$\underline{\lambda} = \{\lambda_1 \geq \cdots \geq \lambda_{n+1}\}$$

$$\pi_{\underline{\lambda}}|_{U(n)} = \sum \pi_{\underline{\mu}},$$

where μ runs over

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1. Multiplicity one.
2. Explicit description depends on a parametrization of all irreducible representations, in this case by the theory of highest weights.

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It may be remarked that a priori the space

$$\mathrm{Hom}_H(\pi_1, \pi_2)$$

may be identically zero, or may be identically infinite dimensional!

Introduction

Branching laws that we consider are for pairs of groups and subgroups which are:

- $GL_{n+1} \supseteq GL_n$
- $SO_{n+1} \supseteq SO_n$
- $U_{n+1} \supseteq U_n$

and some more which go under the name of Bessel subgroup, and Fourier-Jacobi subgroup, but these we will not discuss here.

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Theorem (Aizenbud, Gurevitch, Rallis, Schiffmann)

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Given this theorem, the main question to understand is when

$$\dim \mathrm{Hom}_H(\pi_1, \pi_2) \neq 0.$$

Introduction

Theorem

For π_1 an irreducible admissible generic representation of GL_{n+1}, and π_2 of GL_n, $\dim \mathrm{Hom}(\pi_1, \pi_2) = 1$.

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where the pairs $H' \subseteq G'$ vary over all pure inner forms of a given pair (G, H), and $\Pi_1(G)$ (resp. $\Pi_1(H)$) denotes an L-packet of representions on G (resp. H) which contains a generic representation.

Introduction

For example for unitary groups over reals, we have the pairs:

$$
\begin{aligned}
U(n,0) &\hookrightarrow U(n+1,0) \\
U(n-1,1) &\hookrightarrow U(n,1) \\
&\vdots \\
U(0,n) &\hookrightarrow U(1,n)
\end{aligned}
$$

Review of Local Langlands Correspondence: Harris, Taylor, Henniart, Arthur...

For a reductive algebraic group G over a local field, if $\Pi(G)$ denotes the set of isomorphism classes of representations of G, and $\Sigma(G)$ denotes the set of equivalence classes of (admissible) parameters for G, then there is a surjective map with finite fibers:

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Representations of pure inner forms of G with a given parameter φ are in bijective correspondence with $\widehat{S_\varphi}$, where S_φ denotes the group of connected components of the centralizer of the parameter φ.

The component groups in the cases being considered are elementary abelian 2 groups, i.e., $(\mathbb{Z}/2)^d$, explicitly parametrized by irreducible self-dual summands of the correct parity in the representation

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$$\begin{aligned} S_{\varphi_1} \times S_{\varphi_2} &\to \mathbb{Z}/2 \\ \varphi_{1,i} &\to \varepsilon(\varphi_{1,i} \otimes \varphi_2) \\ \varphi_{2,i} &\to \varepsilon(\varphi_1 \otimes \varphi_{2,i}). \end{aligned}$$

Why care about nontempered branching!

Example 1 (Harder, Langlands, Rapoport):

Let K be a quadratic extension of a number field k, π a cuspidal automorphic representation of $GL_2(\mathbb{A}_K)$ with trivial central character on $\mathbb{A}_k^\times$. Then,

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if and only if $L(s, \mathrm{As}\ \pi)$ has a pole at $s = 1$.

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(i) $\pi = \otimes_v \pi_v$ is locally generic at all the places v;

(ii) $L(s, \mathrm{BC}(\pi))$ has a pole at $s = 1$.

Arthur parameters

The non-tempered representations that we will consider in this lecture are those which arise as local components of automorphic representations, and which are in particular unitary representations. These are parametrized by Arthur by a variant of the Weil-Deligne group:

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$$\psi : W_k' \times SL_2(\mathbb{C}) \to {}^L G$$

where $W_k' = W_k$ or $W_k \times SL_2(\mathbb{C})$ depending on whether k is Archimedean or not, and where ψ restricted to W_k has bounded image in the dual group.

Arthur parameters

Let φ_ψ be the composition:

$$W_k' \to W_k' \times SL_2(\mathbb{C}) \to {}^L G,$$

where the mapping from W_k' to $SL_2(\mathbb{C})$ is given by the diagonal map $(\nu^{1/2}, \nu^{-1/2})$.

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In this lecture we will consider only those representations of $G(k)$ which belong to the L-packet associated to the Langlands parameter ϕ_ψ associated to an A-parameter ψ.

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$$\sigma_2 = \sum_{i=0}^{d} \sigma_{1,i}^{+} \otimes \mathrm{Sym}^{i+1}(\mathbb{C}^2) \oplus \sum_{i=0}^{d} \sigma_{1,i}^{-} \otimes \mathrm{Sym}^{i-1}(\mathbb{C}^2) \oplus \text{ tempered},$$

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then $\dim \mathrm{Hom}(\pi_1, \pi_2) = 1$ for an arbitrary tempered part in σ_2.

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then $\dim \mathrm{Hom}(\pi_1, \pi_2) = 1$ *for an arbitrary tempered part in σ_2. Conversely, if* $\dim \mathrm{Hom}(\pi_1, \pi_2) = 1$*, then the parameters of π_1 and of π_2 can be expressed in this form.*

Remark: The theorem roughly says that any non-tempered part of π_1 corresponding to $\mathrm{Sym}^i(\mathbb{C}^2)$ must have a counterpart either in $\mathrm{Sym}^{i+1}(\mathbb{C}^2)$ or $\mathrm{Sym}^{i-1}(\mathbb{C}^2)$, thus the nontempered part of π_1 determines the nontempered part of π_2 with finite ambiguity.

Some examples for GL_n

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(b)

$$\mathrm{Sym}^{n-2}(\mathbb{C}^2) \oplus \text{ tempered of } GL_2$$

Example 2:

$$\pi_n \otimes \mathrm{Sym}^1(\mathbb{C}^2),$$

a Speh module on $GL_{2n}(k)$ associated to a cuspidal representation π_n of $GL_n(k)$. In this case the only option for σ_2 by our recipe is, Example 2:

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a Speh module on $GL_{2n}(k)$ associated to a cuspidal representation π_n of $GL_n(k)$. In this case the only option for σ_2 by our recipe is,

$$\sigma_2 = \pi_n \oplus \quad \text{arbitrary tempered},$$

so only generic representations appear in this branching.

Comparison with the work of Clozel and Venkatesh

A paper of Clozel [IMRN, 2004] based on elaboration of Arthur's work, and the Burger-Sarnak principle, proves that given a reductive subgroup H of a reductive group G, there is a map from unipotent conjugacy classes in the L-group of G to the unipotent conjugacy classes in the L-group of H which underlies the restriction problem in the unitary case (direct integral and all that!),

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Clozel's theorem has been made precise in some cases by A. Venkatesh [2005]. For example in the restriction problem from $\mathrm{GL}_{n+1}(k)$ to $\mathrm{GL}_n(k)$, if the unipotent element in $\mathrm{GL}_{n+1}(\mathbb{C})$ corresponds to the partition $u = n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$, Clozel's theorem has been made precise in some cases by A. Venkatesh [2005]. For example in the restriction problem from $\mathrm{GL}_{n+1}(k)$ to $\mathrm{GL}_n(k)$, if the unipotent element in $\mathrm{GL}_{n+1}(\mathbb{C})$ corresponds to the partition $u = n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$, then the only unipotent element of $\mathrm{GL}_n(\mathbb{C})$ involved is the one $u^- = n_1 - 1 \geq n_2 - 1 \geq \cdots \geq n_r - 1 \geq 0$, omitting those n_i which are 1, and adding a few 1's at the end if necessary.

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There is an analogous statement for induction of unitary representations of $\mathrm{GL}_n(k)$ to $\mathrm{GL}_{n+1}(k)$.

Comparison with the work of Clozel and Venkatesh

The important point to note is that for both induction and restriction questions in this unitary context, one goes from less tempered to more tempered representations (such as in the Harish-Chandra's Plancherel decomposition for the space $L^2([G \times G]/\Delta(G))$, and in particular, there is no Frobenius reciprocity for unitary representations, whereas we are concerned with admissible representations here which do have Frobenius reciprocity.

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One way to fix this asymmetry, and the corresponding lack of Frobenius reciprocity, is to have the unipotent conjugacy classes u_1, u_2 satisfy,

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Our theorem satisfies these in-equalities.

Classical groups, the local case

Now we discuss branching laws for classical groups emphasizing the case of orthogonal groups. Thus we discuss the branching laws from $SO(n+1)$ to $SO(n)$, more generally from $SO(m)$ to $SO(n)$ with $n+1 \equiv m \bmod 2$ corresponding to Bessel models.

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Let $\psi_1 : W_k' \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathrm{SO}_m$ and $\psi_2 : W_k' \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathrm{SO}_n$ be A-parameters with the corresponding Langlands parameters $\phi_{\psi_1} : W_k' \to {}^L\mathrm{SO}_m$, and $\phi_{\psi_2} : W_k' \to {}^L\mathrm{SO}_n$.

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Let π_1 be an irreducible admissible representation of say $\mathrm{SO}_m(k)$ and π_2 of $\mathrm{SO}_n(k)$ with $m \geq n$ belonging to the L-packets associated to the Langlands parameters $\phi_{\psi_1} : W_k' \to {}^L\mathrm{SO}_m(\mathbb{C})$, and $\phi_{\psi_2} : W_k' \to {}^L\mathrm{SO}_m(\mathbb{C})$.

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The L-groups of the groups $\mathrm{SO}_m(k)$ and $\mathrm{SO}_n(k)$ are the usual orthogonal and symplectic groups which come equipped with their natural representations. When we talk of $L(s, \pi_1 \times \pi_2)$ below, it is for the tensor product of the natural representations of the two L-groups involved. We will also need the adjoint representation of the L-group which is used to define the adjoint L-function.

Conjecture

Let π_1, π_2 be irreducible admissible representations of $\mathrm{SO}_m(k), \mathrm{SO}_n(k)$ belonging to L-packets associated to ϕ_{ψ_1} and ϕ_{ψ_2}, with $m > n$, and $m - n \equiv 1 \bmod 2$. Then if π_2 appears in the Bessel model of π_1,

1. *The Langlands parameters ϕ_{ψ_1} and ϕ_{ψ_2} considered as representations of W_k' inside $\mathrm{GL}_{m'}(\mathbb{C})$ and $\mathrm{GL}_{n'}(\mathbb{C})$ are as in the theorem on $\mathrm{GL}_n(k)$ (the tempered part being arbitrary but of appropriate size).*

Classical groups, the local case

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1. *The Langlands parameters* ϕ_{ψ_1} *and* ϕ_{ψ_2} *considered as representations of* W_k' *inside* $\mathrm{GL}_{m'}(\mathbb{C})$ *and* $\mathrm{GL}_{n'}(\mathbb{C})$ *are as in the theorem on* $\mathrm{GL}_n(k)$ *(the tempered part being arbitrary but of appropriate size).*
2. *If the Langlands parameters* ϕ_{ψ_1} *and* ϕ_{ψ_2} *are as in 1., then the (Vogan) L-packet of representations has a unique member with* $\mathrm{Hom}[\pi_1, \pi_2] \neq 0$*.*

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2. *If the Langlands parameters ϕ_{ψ_1} and ϕ_{ψ_2} are as in 1., then the (Vogan) L-packet of representations has a unique member with $\mathrm{Hom}[\pi_1, \pi_2] \neq 0$.*
3. *The ϵ-factors constructed out of possible symplectic root numbers just as in the earlier works tells which member of the L-packet has the invariant form.*

Remarks: **1.** For representations π_1 and π_2 appearing in the previous conjecture, the L-function

$$\frac{L(s+1/2, \pi_1 \times \pi_2)}{L(s+1, \mathrm{Ad}\,\pi_1)L(s+1, \mathrm{Ad}\,\pi_2)},$$

is not zero (but can have a pole) at $s = 0$.

2. For representations π_1 and π_2 appearing in the previous conjecture for which the A-parameter is discrete, the L-function

$$\frac{L(s+1/2, \pi_1 \times \pi_2)}{L(s+1, \mathrm{Ad}\,\pi_1)L(s+1, \mathrm{Ad}\,\pi_2)},$$

has neither a zero nor a pole at $s = 0$.

An example from the work of Ginzburg, Jiang, Rallis, and Soudry

In a series of paper by Ginzburg, Jiang, Rallis, and Soudry, the authors construct backward lifting from $\mathrm{GL}_n(k)$ to classical groups typically by constructing a representation of a classical group by parabolic induction from the representation of $\mathrm{GL}_n(k)$ which sits as a Levi subgroup, taking its Langlands quotient, and then taking some Bessel or Fourier-Jacobi model (which we will still not define!).

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We describe an instance of their work, and how it fits well with our conjecture.

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The backward lift from $\mathrm{GL}_{2n}(k)$ to $\mathrm{SO}_{2n+1}(k)$ can be constructed as follows. Suppose π is a supercuspidal representation of $\mathrm{GL}_{2n}(k)$ with symplectic Langlands parameter. One induces (a twist of) π from $\mathrm{GL}_{2n}(k)$ which is a Levi subgroup of $\mathrm{SO}_{4n}(k)$ to $\mathrm{SO}_{4n}(k)$, and takes an appropriate Langlands quotient at a point of reducibility, and then compute a Bessel model down to $\mathrm{SO}_{2n+1}(k)$.

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The Langlands parameter of the representation of $\mathrm{SO}_{4n}(k)$ which is a Langlands quotient at a point of reducibility of the principal series representation of $\mathrm{SO}_{4n}(k)$ is,

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In this case, π_2 which is a representation of an odd orthogonal group must have the parameter σ, and so cannot live on a smaller orthogonal group than $\mathrm{SO}_{2n+1}(k)$, and on SO_{2n+1} too, there is no option but to be the backward lift of π_1.

Classical groups, the global case

Here is the conjecture on period integral of Automorphic representations.

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Conjecture

Let F be a number field, and $\Pi_1 \times \Pi_2$ an irreducible automorphic representation of $G = \mathrm{SO}_{n+1}(\mathbb{A}_F) \times \mathrm{SO}_n(\mathbb{A}_F)$ lying in the discrete spectrum, with $H = \mathrm{SO}_n(F)$ a subgroup of $\mathrm{SO}_{n+1}(F)$ defined by a codimension one subspace W of a quadratic space V over F.

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is nonzero for some f an automorphic function on $G(\mathbb{A}_F)$ belonging to $\Pi_1 \times \Pi_2$ if and only if:

1. *The Langlands parameters associated to Π_1 and Π_2 are in the relationship as in the local theorem on GL_n.*

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2. $\mathrm{Hom}_{H(F_v)}[\Pi_{1,v} \otimes \Pi_{2,v}, \mathbb{C}] \neq 0$ *for all places* v *of* F.

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3. $$\frac{L(s+1/2, \Pi_1 \otimes \Pi_2)}{L(s+1, \mathrm{Ad}\,\Pi_1)L(s+1, \mathrm{Ad}\,\Pi_2)},$$

   does not have a zero at $s = 0$.

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2. $\mathrm{Hom}_{H(F_v)}[\Pi_{1,v} \otimes \Pi_{2,v}, \mathbb{C}] \neq 0$ *for all places v of F.*
3. $$\frac{L(s+1/2, \Pi_1 \otimes \Pi_2)}{L(s+1, \mathrm{Ad}\,\Pi_1)L(s+1, \mathrm{Ad}\,\Pi_2)},$$ *does not have a zero at $s = 0$.*

Further, if the L-function condition is satisfied, there is a globally relevant pure inner form G' of G with an automorphic representation $\Pi_1' \otimes \Pi_2'$ nearly equivalent to $\Pi_1 \otimes \Pi_2$ which is globally distinguished by H'.

Theorem

Let F be a number field, and $\Pi_1 \times \Pi_2$ an irreducible automorphic representation of $G = \mathrm{SO}_{n+1}(\mathbb{A}_F) \times \mathrm{SO}_n(\mathbb{A}_F)$ lying in the discrete spectrum, with $H = \mathrm{SO}_n(F)$ a subgroup of $\mathrm{SO}_{n+1}(F)$ defined by a codimension one subspace W of a quadratic space V over F.

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Then if the Langlands parameters associated to Π_1 and Π_2 are in the relationship as in the local theorem on GL_n, then,

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Then if the Langlands parameters associated to Π_1 and Π_2 are in the relationship as in the local theorem on GL_n, then,

$$\frac{L(s+1/2, \Pi_1 \otimes \Pi_2)}{L(s+1, \mathrm{Ad}\,\Pi_1)L(s+1, \mathrm{Ad}\,\Pi_2)},$$

does not have a pole at $s = 0$, and its zeros at $s = 0$ correspond to zeros of $L(1/2, \Pi)$ where Π is a symplectic representation constructed as a tensor product of a subrepresentation of Π_1 with a subrepresentation of Π_2 (self-dual of appropriate parity).

Acknowledgement

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which seems to play a large role in these branching laws came up in the work of **Ichino** and **Ikeda** who proposed that its non-vanishing should control nonvanishing of period integrals.

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2. The initial suggestion to use epsilon factors in these branching laws is due to **Michael Harris**. Thanks Michael!

Thank you!