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17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Sean Howe **Email/Phone:** seanpkh@gmail.com

Speaker's Name: Henri Darmon

Talk Title: Modularity of non-semisimple Galois representations

Date: 12 / 04 / 2014 **Time:** 9 : 30 am / pm (circle one)

List 6-12 key words for the talk: Open Shimura varieties, modularity, p-adic interpolation of L functions, Birch-Swinnerton-Dyer, Bloch-Kato

Please summarize the lecture in 5 or fewer sentences: Begins by describing the connection between "modularity" for open elliptic curves using open modular curves and the rank 1 results on BSD. Then poses the idea of using more general modularity with open Shimura varieties, and more generally by taking p-adic limits of the extensions obtained, to attack BSD. Describes some constructions and results in this direction

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MSRI Workshop:
Automorphic forms, Shimura varieties, Galois representations, and L-functions

Modularity of non-semisimple Galois representations

Henri Darmon

MSRI, Berkeley, December 4, 2014

Credits

This is an account of ideas gleaned in the course of several projects, joint with

Massimo Bertolini

Alan Lauder

Kartik Prasanna

Victor Rotger

Modularity of elliptic curves

Let E be an elliptic curve over $\mathbb{Q}$, of conductor N.

Theorem (Modularity Theorem)

The curve E is uniformised by the modular curve $X_0(N)$.

Theorem (Modularity, cohomological version)

The Galois representation $H^1(E)$ is a quotient of $H^1(X_0(N))$.

Notational convention: $H^i(X) := H^i_{\text{et}}(\overline{X}, \mathbb{Q}_p)(i)$.

$H^1(E) = H^1_{\text{et}}(\overline{E}, \mathbb{Q}_p)(1) =$ Tate module of E.

Modularity of open elliptic curves

Let $E' = E - \Sigma$ be an open subvariety over $\mathbb{Q}$.

$$0 \longrightarrow H^1(E) \longrightarrow H^1(E') \longrightarrow H^0(\Sigma)_0 \longrightarrow 0.$$

Definition (provisional)

The curve E' is said to be *modular* if $H^1(E')$ arises as a subquotient of $H^1(Y)$, where $Y \subset X_0(N)$ is a sub-Shimura variety of $X_0(N)$.

Question: Which E' are modular in this sense?

Open Shimura varieties in $X_0(N)$

Let $\mathcal{O}$:= a quadratic imaginary order.

$\Sigma_{\mathcal{O}} \subset X_0(N) =$ (coarse) moduli space of elliptic curves A with level N structure and an inclusion $\iota : \mathcal{O} \longrightarrow \mathrm{End}(A)$.

$Y_{\mathcal{O}}(N) := X_0(N) - \Sigma_{\mathcal{O}}$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(X_0(N)) & \longrightarrow & H^1(Y_{\mathcal{O}}(N)) & \longrightarrow & H^0(\Sigma_{\mathcal{O}}) & \longrightarrow & 0 \\ & & \downarrow & & \updownarrow & & \uparrow & & \\ 0 & \longrightarrow & H^1(E) & \longrightarrow & H^1(E') & \longrightarrow & H^0(\Sigma) & \longrightarrow & 0. \end{array}$$

The simplest case

Suppose that $\Sigma = \{P_1, P_2\} \subset E(\mathbb{Q})$.

Assume also $H^1(E') \in \mathrm{Ext}^1_{G_\mathbb{Q}}(\mathbb{Q}_p, H^1(E))$ is non-trivial.

Theorem

The following are equivalent:

1. *The curve E' is modular;*
2. $\mathrm{ord}_{s=1} L(E,s) = 1$*;*
3. $E(\mathbb{Q})$ *has rank one and* $Ш(E/\mathbb{Q})$ *is finite;*
4. $\dim_{\mathbb{Q}_p} \mathrm{Ext}^1_{\mathrm{fin}}(\mathbb{Q}_p, H^1(E)) = 1$.

This "modularity result" underlies the proof of BSD in analytic rank ≤ 1.

Sketch of proof

Theorem

The following are equivalent:

1. *The curve E' is modular;*
2. $\mathrm{ord}_{s=1} L(E,s) = 1$;
3. *$E(\mathbb{Q})$ has rank one and $Ш(E/\mathbb{Q})$ is finite;*
4. $\dim_{\mathbb{Q}_p} \mathrm{Ext}^1_{\mathrm{fin}}(\mathbb{Q}_p, H^1(E)) = 1$.

$1 \Rightarrow 2$: Gross-Zagier (1985).

$2 \Rightarrow 3$: Kolyvagin (1989).

$3 \Rightarrow 4$: Trivial.

$4 \Rightarrow 1$: Skinner (2013) [Skinner-Urban + p-adic Gross-Zagier].

The equivariant BSD conjecture

Let ϱ be an Artin representation:

$$\varrho : \mathrm{Gal}(H/\mathbb{Q}) \longrightarrow \mathrm{GL}_n(\mathbb{C}).$$

Hasse-Weil-Artin L-series:

$$L(E, \varrho, s) := \prod_{\ell} \det(1 - \ell^{-s}\, \mathrm{frob}_\ell^{-1} \,|(H^1(E) \otimes \varrho)^{I_\ell})^{-1}.$$

Conjecture (BSD(E, ϱ))

$$\mathrm{ord}_{s=1} L(E, \varrho, s) = \dim_{\mathbb{C}} \hom_{G_{\mathbb{Q}}}(\varrho, E(H) \otimes \mathbb{C}).$$

Ring class characters

Question: Which $\kappa \in \mathrm{Ext}^1(\varrho, H^1(E))$ are realised in

$$0 \longrightarrow H^1(X_0(N)) \longrightarrow H^1(Y_{\mathcal{O}}(N)) \longrightarrow H^0(\Sigma_{\mathcal{O}}) \longrightarrow 0?$$

$$H^0(\Sigma_{\mathcal{O}}) = \oplus_\psi V_\psi, \quad V_\psi := \mathrm{Ind}_K^{\mathbb{Q}} \psi,$$

where ψ are finite order ring class characters.

Let $\kappa \in \mathrm{Ext}^1_{\mathrm{fin}}(V_\psi, H^1(E))$ be a non-trivial extension.

Theorem (Expected?)

The following are equivalent:

1. *The extension κ is modular;*
2. $\mathrm{ord}_{s=1} L(E, V_\psi, s) = 1$;
3. $\dim \mathrm{Ext}^1_{\mathrm{fin}}(V_\psi, H^1(E)) = 1$;
4. $\hom_{G_{\mathbb{Q}}}(V_\psi, E(H_\psi) \otimes \mathbb{C})$ *is one dimensional and the associated Shafarevich-Tate group is finite.*

Modularity

Drawbacks of these modularity results:

1. Very few Artin representations arise in $H^0(\Sigma_{\mathcal{O}})$. So we can't hope to tackle many cases of $\mathrm{BSD}(E, \varrho)$ in this way.
2. The modularity of elements of $\mathrm{Ext}^1_{\mathrm{fin}}(V_\psi, H^1(E))$ is a "rank one phenomenon": if the Ext group has dimension ≥ 2, none of its non-trivial elements are modular!!

Since on some level we hope (expect?) that "every Galois representation arising in geometry ought to be modular", we need to relax the notion of modularity.

Modularity, take 2

First idea: Replace $Y_{\mathcal{O}}(N)$ by (open) Shimura varieties.

Question A: Characterize the non-semisimple Galois representations that are realised in the cohomology of such varieties.

Question B: Suppose that V_1 and V_2 are irreducible Galois representations, and that there is a non-trivial $\kappa \in \mathrm{Ext}^1(V_1, V_2)$ arising from the cohomology of an open Shimura variety. Is $\mathrm{Ext}^1_{\mathsf{fin}}(V_1, V_2)$ necessarily one-dimensional?

If the answer to this question were "yes", it would imply that $E - \{P_1, P_2\}$ is never modular when $\mathrm{rank}(E) \geq 2$.

Modularity, take 3

Second idea: Allow p-adic limits of Galois representations arising in the cohomology of Shimura varieties.

This idea goes back (at least) to the work of Deligne-Serre on Artin representations attached to weight one forms, and is one of the central themes in the subject.

Theorem (Skinner, Urban)

If $L(E, s)$ vanishes to even order ≥ 2 at $s = 1$, then $\dim \mathrm{Ext}^1_{\mathsf{fin}}(\mathbb{Q}_p, H^1(E)) \geq 2$.

The extension classes in this theorem are realised as p-adic limits of (conjecturally semisimple) Galois representations arising in the cohomology of unitary type Shimura varieties.

A p-adic Gross-Zagier formula in rank two

Goals of this lecture:

1. Describe the construction of two *canonical elements* in $\mathrm{Ext}^1_{\mathsf{fin}}(\varrho, H^1(E))$, coming from p-adic limits of modular, geometric, but non-semisimple, Galois representations;
2. Relate these canonical classes to Hasse-Weil-Artin L-functions (both classical, and p-adic);
3. Explain why, in some cases, these classes generate rank two subgroups of the associated Selmer group,

The p-adic Gross-Zagier formula "in rank two" is a linear independence criterion for two canonical Selmer classes, in terms of p-adic L-values.

A geometric construction: the set-up

Let $f \in S_2(\Gamma_0(N))$ be the weight two cusp form attached to E.

Let g and h be modular forms of weight $k \geq 2$ level N_g, N_h, and nebentypes χ_g, χ_h, for which

1. $\gcd(N, N_g N_h) = 1$;
2. $\chi := \chi_g = \chi_h^{-1}$.

Deligne's p-adic representations attached to f, g and h are realised in the middle cohomologies

$$V_f \subset H^1(X_0(N))(-1), \qquad V_g, V_h \subset H^{k-1}(W_k(N_g N_h))(1-k),$$

where $W_k(M)$:= the Kuga-Sato variety fibered over $X_1(M)$.

Some Galois representations

Let

$$V_{gh} := V_g \otimes V_h(k-1).$$

This four-dimensional representation is pure of weight 0 and has determinant 1.

$$V_{fgh} := V_f(1) \otimes V_{gh} = H^1(E) \otimes V_{gh}.$$

This 8-dimensional Galois representation is pure of weight -1 and is isomorphic to its Kummer dual.

Triple product L-functions

Rankin, Garrett, Harris-Kudla: the L-function

$$L(f \otimes g \otimes h, s) := L(V_f \otimes V_g \otimes V_h, s)$$

has analytic continuation and functional equation relating s to $2k - s$, and vanishes at its center of symmetry:

$$L(f \otimes g \otimes h, k) = 0.$$

Beilinson-Bloch:

$$\dim \mathrm{Ext}^1_{\mathrm{fin}}(\mathbb{Q}_p, V_{fgh}) = \mathrm{ord}_{s=k} L(f \otimes g \otimes h, s) \geq 1.$$

In particular, one might hope for a *systematic construction*, analogous to the Heegner point constructions, of such extension classes whose existence is "forced" by a sign in a functional equation.

Gross-Kudla-Schoen cycles

Let $M = \mathrm{lcm}(N, N_g, N_h)$.

The generalised Gross-Kudla-Schoen cycle of weight k is the diagonally embedded

$$W_k(M) \subset X_1(M) \times W_k(M) \times W_k(M).$$

It can be slightly modified so that it becomes null-homologous:

$$\Delta \in \mathrm{CH}^k(X_0(M) \times W_k(M) \times W_k(M))_0.$$

When $k = 2$, work of Gross-Kudla as well as of Xinyi Yuan, Shouwu Zhang and Wei Zhang relates the *Arakelov height* of Δ to $L'(f \otimes g \otimes h, 2)$.

Gross-Kudla-Schoen classes

p-**adic étale Abel-Jacobi map** ($p \nmid M$):

$$\begin{aligned}
&\mathrm{CH}^k(X_0(M) \times W_k(M) \times W_k(M))_0 \\
&\quad \to \quad \mathrm{Ext}^1_{G_\mathbb{Q}}(\mathbb{Q}_p, H^{2k-1}(X_0(M) \times W_k(M) \times W_k(M))(1-k)) \\
&\quad \to \quad \mathrm{Ext}^1_{G_\mathbb{Q}}(\mathbb{Q}_p, H^1(X_0(M)) \otimes H^{k-1}(W_k(M))^{\otimes 2}(1-k)) \\
&\quad \to \quad \mathrm{Ext}^1_{G_\mathbb{Q}}(\mathbb{Q}_p, H^1(E) \otimes V_{gh}).
\end{aligned}$$

Conclusion: When $k \geq 2$, we obtain a global *geometrically modular* class

$$\kappa(f, g, h) \in \mathrm{Ext}^1_{G_\mathbb{Q}}(\mathbb{Q}_p, H^1(E) \otimes V_{gh}),$$

by taking the image of the GKS cycle under the p-adic étale Abel-Jacobi map.

Gross-Kudla-Schoen classes

The $\kappa(f, g, h)$ are not immediately relevant to $BSD(E, \varrho)$ or to Beilinson-Bloch in analytic rank ≥ 2, because

1. The representation V_{gh} is not an *Artin representation*;
2. Results of Gross-Kudla, Yuan-Zhang-Zhang strongly suggest that $\kappa(f, g, h)$ behaves "much like" Heegner points, and should be trivial when $L'(f \otimes g \otimes h, k) = 0$.

These "undesirable" features are not preserved under p-adic limits!

Hida families

Let g and h be classical forms of *weight one* and level M, with associated Artin representations ϱ_g and ϱ_h; let $\varrho_{gh} := \varrho_g \otimes \varrho_h$.

Hecke polynomial for g: $x^2 - a_p(g)x + \chi(p) = (x - \alpha_g)(x - \beta_g)$.

$$g_\alpha := g(z) - \alpha_g g(pz), \qquad g_\beta = g(z) - \beta_g g(pz).$$

Similar notations for h:

$$h_\alpha := h(z) - \alpha_h h(pz), \qquad h_\beta(z) = h(z) - \beta_h h(pz).$$

Let $\underline{g}$ and $\underline{h}$ be Hida families specialising to g_α and h_α respectively in weight 1, and let g_k and h_k denote their higher weight specialisations.

Generalised Kato classes

Definition (Rotger, D)

The *generalised Kato class* attached to (f, g_α, h_α) is the p-adic limit

$$\kappa(f, g_\alpha, h_\alpha) := \lim_{k \longrightarrow 1} \kappa(f, g_k, h_k).$$

The four canonical classes

$$\kappa(f, g_\alpha, h_\alpha), \quad \kappa(f, g_\alpha, h_\beta), \quad \kappa(f, g_\beta, h_\alpha), \quad \kappa(f, g_\beta, h_\beta)$$

belong to $\mathrm{Ext}^1_{G_{\mathbb{Q}}}(\mathbb{Q}_p, H^1(E) \otimes V_{gh})$, and should carry information about $BSD(E, V_{gh})$.

Explanation for the terminology

$\kappa(f, g_\alpha, h_\alpha)$ can also be defined when g_α and/or h_α is an Eisenstein series rather than a cusp form.

1. When g_α and h_α are both Eisenstein series of weight one, by taking p-adic limits of classes $\kappa(f, g_2, h_2)$, defined from p-adic étale regulators of distinguished elements in $\mathrm{CH}^2(X_1(M), 2)$: the *Beilinson-Kato elements*;
2. When g_α is a cusp form and h_α is an Eisenstein series, by taking p-adic limits of classes $\kappa(f, g_2, h_2)$, defined from p-adic étale regulators of elements in $\mathrm{CH}^2(X_1(M)^2, 1)$: the *Beilinson-Flach elements*.

Both Beilinson-Kato and Beilinson-Flach elements can be viewed as "degenerate cases" of diagonal cycles in $\mathrm{CH}^2(X_1(M)^3)$.

The first reciprocity law

The classes $\kappa(f, g_\alpha, h_\alpha)$ arise as p-adic limits of cristalline extensions, but there is no reason for them to be cristalline.

Theorem (Rotger, D)

The classes $\kappa(f, g_\alpha, h_\alpha), \dots$ are non-cristalline if and only if $L(f \otimes g \otimes h, 1) \neq 0$.
Assume further that $\varrho_{gh}(\mathrm{frob}_p)$ has distinct eigenvalues. Then the four generalised Kato classes are linearly independent and their images generate the "singular quotient"

$$\frac{\mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\mathbb{Q}_p, V_{fgh})}{\mathrm{Ext}^1_{\mathrm{fin}}(\mathbb{Q}_p, V_{fgh})}.$$

Application to $BSD(E, \varrho_{gh})$

Theorem (Rotger, D)

If $L(E, \varrho_{gh}, 1) \neq 0$, *then* $\hom_{G_{\mathbb{Q}}}(\varrho_{gh}, E(H) \otimes \mathbb{C}) = 0$.

Kato's setting: when $g = E_1(\chi_1, \chi_2)$ and $h = E_1(1, (\chi_1\chi_2)^{-1})$, then

$$\varrho_{gh} = \chi_1 + \chi_2 + \bar{\chi}_1 + \bar{\chi}_2,$$

and one recovers

Theorem (Kato)

If $L(E, \chi_1, 1) \neq 0$, *then* $\hom_{G_{\mathbb{Q}}}(\chi_1, E(H) \otimes \mathbb{C}) = 0$.

In this setting, Kato also obtains the finiteness of the relevant components of the Shafarevich-Tate group using "tame deformations" of the Kato classes.

The Beilinson-Flach setting

When g is cuspidal and $h = E_1(1, \chi^{-1})$, then

$$\varrho_{gh} = \varrho_g \oplus \bar{\varrho}_g,$$

and one obtains

Theorem (Bertolini, Rotger, D)

If $L(E, \varrho_g, 1) \neq 0$, then $\hom_{G_{\mathbb{Q}}}(\varrho_g, E(H) \otimes \mathbb{C}) = 0$.

These ideas have been taken up and developped a lot further by Kings, Lei, Loeffler, Zerbes, leading (notably) to a proof of the finiteness of certain p-parts of the Shafarevich-Tate group, Iwasawa main conjectures, etc.

See Sarah's lecture this afternoon...

The first reciprocity law: idea of the proof

Key ingredient: the Hida-Harris-Tilouine p-adic L-function, interpolating the *central critical values* $L(f_k, g_\ell, h_m, c)$.

Definition

A triple (k, ℓ, m) of weights is said to be *balanced* if neither weight is $\geq$ than the sum of the other two. Otherwise, the triple is said to be *unbalanced*, and the largest weight is called the *dominant weight*.

$\Sigma_{\mathrm{bal}} := \{(k, \ell, m)$ such that (k, ℓ, m) is balanced $\} \subset (\mathbb{Z}^{\geq 1})^3$;

$\Sigma_f := \{(k, \ell, m)$ such that k is the dominant weight.$\}$;

$\Sigma_g := \{(k, \ell, m)$ such that ℓ is the dominant weight.$\}$;

$\Sigma_h := \{(k, \ell, m)$ such that m is the dominant weight.$\}$.

The three Hida-Harris-Tilouine L-functions

Note that $L(f_k \otimes g_\ell \otimes h_m, c) = 0$ for all $(k, \ell, m) \in \Sigma_{\mathrm{bal}}$.

$L_p^f(\underline{f}, \underline{g}, \underline{h})$: interpolates $\frac{L(f_k, g_\ell, h_m, c)}{\langle f_k, f_k \rangle^2}$ as $(k, \ell, m) \in \Sigma_f$;

$L_p^g(\underline{f}, \underline{g}, \underline{h})$: interpolates $\frac{L(f_k, g_\ell, h_m, c)}{\langle g_\ell, g_\ell \rangle^2}$ as $(k, \ell, m) \in \Sigma_g$;

$L_p^h(\underline{f}, \underline{g}, \underline{h})$: interpolates $\frac{L(f_k, g_\ell, h_m, c)}{\langle h_m, h_m \rangle^2}$ as $(k, \ell, m) \in \Sigma_h$.

A p-adic Gross-Kudla formula

Let $(k, \ell, m) \in \Sigma_{\mathrm{bal}}$. $\quad D_{fgh} := (V_{fgh} \otimes B_{\mathrm{cris}})^{G_{\mathbb{Q}_p}}$.

$\log_p : \mathrm{Ext}^1_{G_{\mathbb{Q}_p},\mathrm{fin}}(\mathbb{Q}_p, V_{fgh}) \longrightarrow \mathrm{Fil}^0(D_{fgh})^\vee$.

$\mathrm{Fil}^0 D_{fgh} = \langle \omega_f\omega_g\omega_h, \quad \eta_f\omega_g\omega_h, \quad \omega_f\eta_g\omega_h, \quad \omega_f\omega_g\eta_h \rangle$.

Theorem (Rotger, D)

In the balanced region,

$$\begin{aligned}
\log_p(\kappa(f,g,h))(\eta_f\omega_g\omega_h) &= (*) \times L_p^f(f,g,h);\\
\log_p(\kappa(f,g,h))(\omega_f\eta_g\omega_h) &= (*) \times L_p^g(f,g,h);\\
\log_p(\kappa(f,g,h))(\omega_f\omega_g\eta_h) &= (*) \times L_p^h(f,g,h).
\end{aligned}$$

This formula was inspired by a p-adic Gross-Zagier formula for Heegner points (Bertolini, Prasanna, D).

The first reciprocity law

Let $(k, \ell, m) = (2, 1, 1) \in \Sigma_f$.

$\exp^* : \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\mathbb{Q}_p, V_{fgh}) / \mathrm{Ext}^1_{\mathrm{fin}} \longrightarrow \mathrm{Fil}^0 D_{fgh} = \mathrm{Fil}^0(D_f) \otimes D_{gh}$.

Theorem (Rotger, D)

$\exp_p^*(\kappa(f, g_\alpha, h_\alpha)) \sim L_p^f(f, g, h) \cdot \omega_f \eta_g^\alpha \eta_h^\alpha \sim L(f, g, h, 1)$.

Proof: Perrin-Riou:

$$\lim_{(k,\ell,m)\longrightarrow(2,1,1)} \log_p(\kappa(f_k, g_\ell, h_m))(\eta_{f_k}\omega_{g_\ell}\omega_{h_m}) \sim \exp^*(\kappa(f, g_\alpha, h_\alpha)).$$

Hence

$$\lim_{(k,\ell,m)\longrightarrow(2,1,1)} p\text{-adic Gross-Kudla}(\eta_{f_k}\omega_{g_\ell}\omega_{h_m}) = \text{First reciprocity law}.$$

The second reciprocity law

If $L(f, g, h, 1) = 0$, then the four classes

$$\kappa(f, g_\alpha, h_\alpha), \quad \kappa(f, g_\alpha, h_\beta), \quad \kappa(f, g_\beta, h_\alpha), \quad \kappa(f, g_\beta, h_\beta)$$

belong to $\mathrm{Sel}_p(E, \varrho_{gh}) \subset H^1(\mathbb{Q}, H^1(E) \otimes V_{gh})$.

Furthermore, $\mathrm{ord}_{s=1} L(E, \varrho_{gh}, s) \geq 2$.

Conjecturally, $\mathrm{Sel}_p(E, \varrho_{gh})$ has rank ≥ 2.

The first reciprocity law led to insights for $\mathrm{BSD}(E, \varrho_{gh})$ in rank zero settings; the second reciprocity law should do something similar in this rank two setting, by relating

$$\log_{\alpha\beta}(\kappa(f, g_\alpha, h_\alpha)) := \log_p(\kappa(f, g_\alpha, h_\alpha)(\omega_f \eta_g^\alpha \eta_h^\beta)$$

to p-adic L-values.

A plethora of p-adic L-functions

There are in fact 12 relevant p-adic L-functions!!

$$L_p^f(\underline{f}, \underline{g}_\alpha, \underline{h}_\alpha), \quad L_p^f(\underline{f}, \underline{g}_\alpha, \underline{h}_\beta), \quad L_p^f(\underline{f}, \underline{g}_\beta, \underline{h}_\alpha), \quad L_p^f(\underline{f}, \underline{g}_\beta, \underline{h}_\beta);$$

$$L_p^g(\underline{f}, \underline{g}_\alpha, \underline{h}_\alpha), \quad L_p^g(\underline{f}, \underline{g}_\alpha, \underline{h}_\beta), \quad L_p^g(\underline{f}, \underline{g}_\beta, \underline{h}_\alpha), \quad L_p^g(\underline{f}, \underline{g}_\beta, \underline{h}_\beta);$$

$$L_p^h(\underline{f}, \underline{g}_\alpha, \underline{h}_\alpha), \quad L_p^h(\underline{f}, \underline{g}_\alpha, \underline{h}_\beta), \quad L_p^h(\underline{f}, \underline{g}_\beta, \underline{h}_\alpha), \quad L_p^h(\underline{f}, \underline{g}_\beta, \underline{h}_\beta);$$

At (f, g, h) of weight $(2, 1, 1)$, there are 5 relevant values:

$$L_p^f(f, g_\alpha, h_\alpha) \sim L_p^f(f, g_\alpha, h_\beta) \sim \cdots \sim L(f, g, h, 1);$$

$$L_p^g(f, g_\alpha, h_\alpha) = L_p^g(f, g_\alpha, h_\beta), \qquad L_p^g(f, g_\beta, h_\alpha) = L_p^g(f, g_\beta, h_\beta);$$

$$L_p^h(f, g_\alpha, h_\alpha) = L_p^h(f, g_\beta, h_\alpha), \qquad L_p^h(f, g_\alpha, h_\alpha) = L_p^h(f, g_\beta, h_\alpha).$$

The second reciprocity law

Theorem (Rotger, D)

Let $L \in \mathbb{C}_p$ be given by $L^2 \sim L_p^g(f, g_\alpha, h)$.

$$\log_{\alpha\beta}(\kappa(f, g_\alpha, h_\alpha)) = L, \quad \log_{\alpha\alpha}(\kappa(f, g_\alpha, h_\alpha)) = 0,$$

$$\log_{\alpha\beta}(\kappa(f, g_\alpha, h_\beta)) = 0, \quad \log_{\alpha\alpha}(\kappa(f, g_\alpha, h_\beta)) = L.$$

This reciprocity law has been arranged as a 2×2 matrix of "p-adic Gross-Zagier formulae", involving a single L-value but two Selmer classes.

It can be viewed as a Gross-Zagier formula in rank two.

A p-adic Gross-Zagier formula in rank two

Corollary (Rotger, D)

If $L_p^g(f, g_\alpha, h) \neq 0$*, then the classes* $\kappa(f, g_\alpha, h_\alpha)$ *and* $\kappa(f, g_\alpha, h_\beta)$ *are linearly independent in* $\mathrm{Sel}_p(E, \varrho_{gh})$*.*

Lauder's algorithms enable the efficient numerical calculation of $L_p^g(f, g_\alpha, h)$.

Many cases where the generalised Kato classes generate a rank two subgroup of the Selmer group have thus been exhibited experimentally.

Summary

- The p-adic world has rich features (notably thanks to the possibility of p-adic variation), opening up the possibility of Gross-Zagier formulae for elliptic curves of rank > 1.
- Things are much less clear in the archimedean setting.
- Theorems about Selmer groups are very nice ..... theorems about Mordell-Weil or Shafarevich-Tate are rare and sublime. Rank two Gross-Zagier is not one of these!!

Happy Birthday Michael!