MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Sean Howe **Email/Phone:** seanpkh@gmail.com

Speaker's Name: Sarah Zerbes

Talk Title: Euler Systems

Date: 12 / 4 / 2014 **Time:** 2 : 00 am / pm (circle one)

List 6-12 key words for the talk: Euler-systems, Bloch-Kato, p-adic L-Functions, Hida Families, Families of Galois representations

Please summarize the lecture in 5 or fewer sentences: Describes recent work on Euler-systems for non-effective motives and Families of Euler-systems. Generalizes conjectures and results from effective case (e.g. Euler-system machine) and provides many new examples.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Euler Systems

Zerbes 1

(joint w/ D. Loeffler)

I. A conjecture of Perrin-Riou

M = motive / $\mathbb{Q}$ $\quad L(M,s)$ = L-function of M.

Assume $L(M,s)$ has a meromorphic continuation and a functional equation.

M_p = p-adic realization of M + action of $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

$H^1_f(\mathbb{Q}, M_p)$ = BK - Selmer group $\subseteq H^1(\mathbb{Q}, M_p)$ cut out by local conditions.

($\ell \neq p$ unramified local condition; at p H^1_f - local condition)

<u>Conjecture</u> (Bloch-Kato)

$$\mathrm{ord}_{s=0} L(M,s) = \dim H^1_f(\mathbb{Q}, M_p^*(1)) - \dim H^0(\mathbb{Q}, M_p^*(1))$$

<u>Def'n</u> M is effective if all Hodge numbers are ≥ 0

<u>Ex</u>: $M = \mathbb{Q}$ or $M = M(f)$ are effective, $\mathbb{Q}(1)$ is not effective.
motive attached to mod form of wt. ≥ 2

Facts if M effective, $M \neq \mathbb{Q}$

- Local H^1_f condition at p - relaxed local condition (if local L-factor doesn't vanish)
- $\mathrm{ord}_{s=0} L(M,s) = \dim (M_{\mathrm{betti}})^{c=1}$ $\quad c$ = complex conj.

$$\underset{\text{(Bloch-Kato)}}{\Longrightarrow} \dim H^1_f(\mathbb{Q}, M_p^*(1)) = \dim M_{\mathrm{Betti}}^{c=1} = d^+$$

Conjecture (Perrin-Riou): If M is effective, $\exists$ a non-zero system of elements $z_m \in \Lambda^{d^+} H^1(\mathbb{Q}(\mu_m)^+, M_p^*(1))$

s.t. $\mathrm{cores}^{\mathbb{Q}(\mu_{m\ell})^+}_{\mathbb{Q}(\mu_m)^+} z_{m\ell} = \begin{cases} z_m & \text{if } \ell \mid m \text{ or } M_p^*(1) \text{ ram. at } \ell \\ P_\ell(\sigma_\ell^{-1}) z_m & \end{cases}$

where $P_\ell(X) = \det(1 - \sigma_\ell^{-1} X \mid M)$, σ_ℓ^{-1} = arithmetic Frob.

Def'n: Such a system is a rank d^+ Euler system for M.

Remark: If M/K, K a # field, an Euler system requires classes over all abelian extensions of K.

Thm (Kolyvagin, Rubin, Perrin-Riou) If such a rank d^+-ES exists, and $z_1 \neq 0$, then (under technical hypotheses), $H^1_F(\mathbb{Q}, M_p^*(1))$ is d^+-dimensional.

(ES machine).

Remark Important that M effective for relaxed local condition at p.

Known cases ($d^+ = 1$)

* $M = \mathbb{Q}$ cyclotomic units
* $M = M(f)$, f mod. form of wt ≥ 2 (Kato ES)
* $M =$ imag. quadratic field, elliptic units

Problem – no non-trivial examples when $d^+ > 1$.

II A new Euler-System

Theorem (Lei-LZ, Kings-LZ) Let f, g be modular forms of weights $k+2$, $k'+2 \geq 2$

$0 \leq j \leq \min(k, k')$

$\exists$ a rank 1 ES for $M = M(f) \otimes M(g)(1+j)$ related to $L_p(f, g, 1+j)$

Fact
* $d^+(M) = 2$, but M is not effective
  (Hodge no.s $-1-j$, $k'-j$, $k-j$, $k+k'+1-j$)
* $\mathrm{ord}_{s=0} L(M, s) = \mathrm{ord}_{s=1-j} L(f, g, s) = 1$
* H^1_f local condition at p is not the relaxed one.

Def'n For $r \geq 0$, M is <u>r-critical</u> if Archimedean Γ-factor $L_\infty(M,s)$ has a pole at $s=0$ of order r, $L_\infty(M^*(1),0) \neq \infty$

Remarks (i) M r-critical $\Leftrightarrow$ $d^+ - r$ Hodge #s are <0

(ii) 0-critical = Deligne's definition of critical

(iii) M r-critical $\Rightarrow$ $\mathrm{ord}_{s=0} L(M,s) \geq r$

Conjecture 1: If M is r-critical then $\exists$ a non-zero ES of rank r, (z_m), $z_m \in \Lambda^r H^1_f(\mathbb{Q}(\mu_m)^+, M_p^*(1))$

Remark: $r = d^+$ is Perrin-Riou's conjecture

Theorem (LZ): If M is 1-critical and (z_m) is a rank 1 ES w/ $z_1 \neq 0$, under technical hypotheses

$$H^1_f(\mathbb{Q}, M_p^*(1)) \text{ is 1-dim'l}$$

Remark Need to adapt Euler System machine to take into account the non-relaxed H^1_f local condition.

Examples (i) $M = M(f) \otimes M(g)(1+j)$ $\quad 0 \leq j \leq \min(k,k')$

(ii) $M = M_{Asai}(f)(1+j)$ $\quad f$ quad. Hilbert modular form of weights $k+2, k'+2$, $0 \leq j \leq \min(k,k')$

(iii) $M = M_{spin}(f)(1+j)$ $\quad f$ genus 2 Siegel modular form wts. $(k+3, k'+3)$ $\quad k \geq k' \geq 0 \quad 0 \leq j \leq k'$

iv) $M(1)$, $M \subset h^2(Sh(U(2,1)))$ rank 3 motive / imag. quad field

Rank 1 ES

(i) LLZ, KLZ
(ii) } In progress — Lei - LZ
(iii) } In progress — Lemma - LZ
(iv) } In progress — Skinner - LZ

To get at 0-critical motives, use p-adic deformation

III Euler systems in p-adic families

Def: A = complete local $\mathbb{Z}_p$-algebra $X = \mathrm{Spf} A$

A family of motivic Galois rep'ns is a finite free A-mod V with A-linear $G_{\mathbb{Q}}$-action s.t. $\forall x \in$ Zariski dense set of points X_{cl},

$$V_x \cong (M_x)_p, \qquad M_x = \text{motive}.$$

Key example: cyclotomic deformation of M_p.

$A = \Lambda(\mathbb{Z}_p^\times)$

$V = M_p \otimes A$ w/ g action on A $g \in G_{\mathbb{Q}}$ given by canonical character.

V specializes to $M_p(j)$ $\forall j \in \mathbb{Z}$.

Def r-refinement of V = a direct summand $W \subseteq V$ stable under $G_{\mathbb{Q}_p}$ s.t. $\forall x \in$ Zariski-dense set,

$X_{cl,W} \subseteq X_{cl}$

- M_x is r-critical
- W_x has all HT weights ≥ 1, HT weights of V_x / W_x are ≤ 0

<u>Conj. 2</u> V = Family of motivic Galois representations, $W \subseteq V$ refinement.

$\exists$ non-trivial rank r ES (z_m)

$$z_m \in \Lambda^r (H^1(\mathbb{Q}(\mu_m)^+, V))$$

s.t. • $\mathrm{loc}_p(z_m) \in \mathrm{im}\, \Lambda^r H^1(\mathbb{Q}_p(\mu_m)^+, W)$

• $\forall\ x \in X_{cl,w}$, $(z_m)_x$ should be ES from conj. 1

<u>Remark</u> $\forall\ x \in X_{cl,w}$ (generically)

$$H^1_f(\mathbb{Q}_p(\mu_m)^+, V_x) = \mathrm{im}\, H^1(\mathbb{Q}_p(\mu_m)^+, W_x)$$

$$\Rightarrow (z_m) \in \Lambda^r H^1_f(\mathbb{Q}_p(\mu_m)^+, V_x)$$

<u>Examples</u> Rankin-Selberg convolutions

F, g Hida Families, $\exists$ 3-parameter Family of $G_{\mathbb{Q}}$-reps

$$V(F)^* \hat{\otimes} V(g)^* \otimes \Lambda(\mathbb{Z}_p^\times)$$

Interpolating $M_p(f_k)^* \otimes M_p(g_{k'})^*(1+j)$

for varying k, k', j

2-refinement $W_2 = \{0\}$ $X_{cl,W_2} = \{k, k' \geq 0, j \leq -1\}$

1-refinement $W_1 = \mathcal{F}^+ V(F)^* \hat{\otimes} \mathcal{F}^+ V(g)^* \hat{\otimes} \Lambda(\mathbb{Z}_p^\times)$

$$X_{cl,W_1} = \begin{cases} k, k' \geq 0 \\ 0 \leq j \leq \min(k, k') \end{cases}$$

0-refinements $W_{0a} = \mathcal{F}^+ V(F)^* \hat{\otimes} V(g)^* \hat{\otimes} \Lambda(\mathbb{Z}_p^\times)$

$$X_{cl,W_{0a}} = \{k'+1 \leq j \leq k\}$$

similar for W_{0b}

Thm (KLZ): (I) Our ES for $M(f_K) \otimes M(g_{K'})(1-j)$ interpolate along W_1

(II) (Explicit recip. law)

$\exists$ rank lowering operators corresponding to

$$W_1 \hookrightarrow \begin{cases} W_{0a} \\ W_{0b} \end{cases}$$

mapping the rank 1 ES to Hida's 2 p-adic L-functions

(rank-lowering operators = generalizations of P-R regulator map (LZ))

(III) If rank 2 ES exists, it maps to our ES under rank lowering operators.

---

general expectation $W' \subseteq W$ r', r refinements of V,

$\exists$ rank-lowering operator

$$\Lambda^r H^1_f(\mathbb{Q}(\mu_m)^+, V) \to \Lambda^{r'} H^1_f(\mathbb{Q}(\mu_m)^+, V)$$

compatible w/ ES.

In progress $\overset{(KLZ)}{\sim}$ replace Hida families by Coleman families

replace subrep by sub-Robba ring (φ, Γ)-module.