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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Sean Howe **Email/Phone:** seanpkh@gmail.com

Speaker's Name: Gaetan Chenevier

Talk Title: On conductor 1 algebraic automorphic representations of $GL(n)$ over $\mathbb{Q}$, and applications

Date: 12 / 05 / 2014 **Time:** 10 : 30 [✓ am] / pm (circle one)

List 6-12 key words for the talk: Automorphic forms, unramified representations, counting, explicit functoriality, Fontaine-Mazur

Please summarize the lecture in 5 or fewer sentences: Explains some results on classifying and counting everywhere unramified algebraic automorphic representations (and thus conjecturally certain Galois representations). Gives applications to computing zeta functions and to explicitly computing functorial transfers from $SO(n)$ to $GL(n)$.

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On algebraic automorphic representations of conductor 1

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Plan

1. GL_n case
2. General G

Automorphic representations

Consider cuspidal automorphic rep's. π of $\mathrm{GL}_n(\mathbb{A})$ such that :

(i) π_p is unramified for each prime p.

(ii) π_∞ is algebraic, i.e. $\inf \pi_\infty \ \subset \mathrm{M}_n(\mathbb{C})$ has integer eigenvalues

$$k_1 \geq k_2 \geq \cdots \geq k_n,$$

called the weights of π.

Set also $\mathrm{w}(\pi) = k_1 - k_n$: motivic weight of (effective twist of) π.

General problem : Can we classify those π ?

A motivation : galois representations

Fix a prime ℓ and an embedding $\iota : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$.

Consider irred. cont. rep's $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ such that :

(i) ρ is unramified outside ℓ,

(ii) ρ is crystalline at ℓ.

Conjecture (Langlands-Fontaine-Mazur) : *There exists a unique bijection $\pi \mapsto \rho_\pi$ such that for all primes $p \neq \ell$*

$$\det(T - \rho_\pi(\mathrm{Frob}_p)) = \iota \det(T - \mathrm{c}(\pi_p)).$$

Moreover, the weights of π are the Hodge-Tate weights of ρ_π at ℓ.

Regular case : $\pi \mapsto \rho$ has been defined and some properties proved.

A counting problem

$\mathrm{N}(k_1, \cdots, k_n) =$ number of π of weights $k_1 \geq \cdots \geq k_n$.

Finite by general results of Harish-Chandra.

e.g. $\mathrm{N}(k) = 1, \quad \mathrm{N}(k, 0) = \dim \mathrm{S}_{k+1}(\mathrm{SL}_2(\mathbb{Z}))$,
... no result for any $n > 2$ valid for all weights.

Problems : – no discrete series for $\mathrm{SL}_n(\mathbb{R})$ for $n > 2$,
– difficult to compute geometric side of trace formula.

Essentially-self-dual, regular, case

$\mathrm{N}^{\perp}(k_1, \cdots, k_n) =$ subnumber of π of such that $\pi^{\vee} \simeq \pi|\cdot|^{-k_1-k_n}$.

Theorem (Ch.-Renard, Taïbi) *Explicit given formula for* $\mathrm{N}^{\perp}(k_1, \cdots, k_n)$ *valid for all* n *and all* $k_1 > \cdots > k_n$*, implemented on a computer for* $n \leq 15$ *(so far). Conditional if two* k_i*'s are consecutive.*

Basic idea of proofs. Induction on n. Such a π descends to a collection of aut. rep. of classical groups over $\mathbb{Z}$ such that π_∞ discrete series (Arthur). Compute (or use known) dim. of certain spaces of aut. rep. of classical groups, and substract "endoscopic contributions" (Arthur's Multiplicity Formula). Condition "unramified everywhere" important in A.M.F.

Dim. formula previously known for Sp_{2g} with $g \leq 2$ (Igusa, Tsushima), get split SO_m with $m \leq 5$ by exc. isogenies.

Ch.-Renard : proof theorem for $n \leq 5$, + conditionnally $n = 6$ and 7 (use $\mathbb{R}$-anisotropic inner forms to compute dim.).

Taïbi : general case. He found an algorithm to compute the "Euler characteristic of discrete spectrum of split classical groups for any cohomological weight", starting form Arthur's formula in his paper L^2-*Lefschetz..* As an application, he deduces dimension spaces of vector valued Siegel cusp forms for $\mathrm{Sp}_{2g}(\mathbb{Z})$ for $g \leq 7$.

An (unexpected) application

Applies conjecturally to $\zeta(s, \mathcal{M}_{g,n})$.

Thanks to works of Bergström, Faber, van der Geer & Megarbané, it led to a conjectural purely automorphic expression for this ζ function for

$$(g, n) = (3, 17).$$

Two interesting π's of dimension 6 and weight $23 = \dim \mathcal{M}_{3,17}$ occur (there are 7 such ess. self. dual reg. $\pi's$).

First case where $\zeta(s, \mathcal{M}_{g,n})$ not entirely explained by Siegel modular forms of genus $\leq g$.

A different method/result

Theorem : π *cusp. aut. rep. of* $\mathrm{GL}_n(\mathbb{A})$ *satisfying (i) and (ii).*
Assume $\mathrm{w}(\pi) \leq 20$*. Then :*

(a) *either* $n \leq 2$,

(b) *or* π *is a twist of the unique rep. of* $\mathrm{GL}_4(\mathbb{A})$ *sat. (i) and (ii) and with weights* $19 \geq 13 \geq 6 \geq 0$.

Known to Serre and Mestre that for $\mathrm{w}(\pi) < 11$ then $n = 1$.

Sketck of proof

Idea in the continuation of ideas of Stark, Odlyzko, Serre, Mestre, Miller : contradict existence of π using analytic properties of $\Lambda(s, \pi \times \pi^\vee)$, using Riemann-Weil explicit formula.

Let $W = \mathrm{W}_\mathbb{R}/\mathbb{R}_{>0}$ (an extension of $\mathbb{Z}/2\mathbb{Z}$ by S^1).

K = Grothendieck ring of $\mathbb{C}$-rep. of $W = \mathbb{Z} \oplus \mathbb{Z}\varepsilon \bigoplus_{w>0} \mathbb{Z}\,\mathrm{I}_w$.

If π is the unitary twist of a Π satisfying (i) and (ii), then $\mathrm{L}(\pi_\infty) \in K$ (Clozel purity lemma).

Choice of test function : $F : \mathbb{R} \to \mathbb{R}$, even, $\mathcal{C}^2$, compact support. And define $\Phi(s) = \int_\mathbb{R} F(t) e^{(s-1/2)t} dt$, $s \in \mathbb{C}$.

Linear map $\mathrm{J}_F : \mathrm{K} \to \mathbb{R}$, $\qquad W \mapsto -\frac{1}{2\pi i} \int_{\mathrm{Re}(s)=1/2} \frac{\Gamma'}{\Gamma}(W, s)\Phi(s) ds.$

The explicit formula (following Mestre)

Fix Π, Π' sat. (i) and (ii), let π and π' be their unitary twists.

Result of a contour integration $\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Lambda'}{\Lambda}(s, \pi \times \pi')\Phi(s)ds$.

Set $V = \mathrm{L}(\pi_\infty)$ and $V' = \mathrm{L}(\pi'_\infty)$.

$$\sum_{\mu} \Phi(\mu) - 2\,\delta_{\pi', \pi^\vee}\ \Phi(0)$$

$$=$$

$$-2\,\mathrm{J}_F(V \otimes V') - 2\,\mathrm{Re}\ \sum_{p^k} \mathrm{trace}(\mathrm{c}(\pi_p)^k \otimes \mathrm{c}(\pi'_p)^k) F(k\log(p)) \frac{\log(p)}{p^{k/2}}.$$

The inequality

Assume $F \geq 0$, $\mathrm{Re}\,\Phi(s) \geq 0$ if $0 \leq \mathrm{Re}\,s \leq 1$, and $\pi' = \pi^\vee = \overline{\pi}$.

(e.g. $F = f(x/\lambda)/\mathrm{ch}(x/2)$ where f is Odlyzko function and $\lambda > 0$.)

Then $\mathrm{J}_F(V^2) \leq \Phi(0)$.

Proposition : For F well chosen, the quadratic form $W \to \mathrm{J}_F(W^2)$, $\mathrm{K} \to \mathbb{R}$, is positive definite on $\mathrm{K}^{\leq 20}$.

Proof : a Gram matrix computation using Odlyzko function ($\lambda = \log 9$) !

Corollary : Only finitely many possible π_∞, hence π !

List all possible V with the computer. Get Thm. when $\mathrm{w}(\pi) \leq 17$. A few possible V in general, *regular* and with $\dim V \leq 5$ in all cases. Conclude if π is selfdual.

End of proof

Show that when π exists, then there is a unique one.

Observation of Taïbi : let $\pi_1, \ldots, \pi_k$ be k different aut. rep. of $\mathrm{GL}_n(\mathbb{A})$ such that $\mathrm{L}(\pi_\infty) \simeq V$. Then :

$$\mathrm{J}_F(V^2) \leq \frac{1}{k}\Phi(0)$$

Proof : same as before applied to $\Lambda(s, (\oplus_i \pi_i) \otimes (\oplus_i \pi_i^\vee))$.

Check that for all previously found V then $k \leq 1$. $\square$

Arthur-Langlands conjecture

G reductive gp. scheme over $\mathbb{Z}$

$\widehat{G}$ = red. group over $\mathbb{C}$ dual to $G(\mathbb{C})$

$\quad$ = Langlands dual of $G_{\mathbb{Q}}$ (Gross)

π discrete aut. rep. of $G(\mathbb{A})$ s.t. $\pi_p^{G(\mathbb{Z}_p)} \neq 0$ for all p.

$\rho : \widehat{G} \to \mathrm{GL}_n(\mathbb{C})$.

Conjecture : $\exists k \geq 1$, *and for* $i = 1, \ldots, k$, *integers* d_i, $n_i \geq 1$, *and a cusp. aut. rep.* π_i *of* $\mathrm{GL}_{n_i}(\mathbb{A})$, *such that :*

(a) $n = \sum_{i=1}^{k} d_i n_i$,

(b) $\mathrm{L}(s, \pi, \rho) = \prod_i \prod_{j=0}^{d_i - 1} \mathrm{L}(s + j - \frac{d_i - 1}{2}, \pi_i)$,

(c) $\rho(\inf\, \pi_\infty) = \oplus_i \inf\, \pi_i \otimes \mathrm{diag}(\frac{d_i - 1}{2}, \ldots, \frac{1 - d_i}{2})$.

If conjecture holds for (π, ρ), write $\psi(\pi, \rho) = \oplus_i \pi_i[d_i]$.

Two remarks :

(i) If $\langle \mathrm{inf}\pi_\infty, \alpha \rangle \in \mathbb{Z}$ for each root α of $\widehat{G}$, then π_i is algebraic up to twist for each i.

(ii) If $\rho^\vee \simeq \rho$, then π_i selfdual for each i.

Arthur's theorem : *Conjecture holds for* (π, St) *if* $G = \mathrm{Sp}_{2g}$ *or split* SO_m *over* $\mathbb{Z}$. *+ Converse result (Arthur's multiplicity formula).*

Example $G = \mathrm{PGSp}_4 = \mathrm{SO}_5$

$\widehat{G} = \mathrm{Sp}_4(\mathbb{C})$

Fix $w > v > 0$ odd integers.

The number of cuspidal π of G such that π_∞ hol. discrete series of inf. car. $\mathrm{diag}(\frac{w}{2}, \frac{v}{2})$ (with mult.)

= dimension of a certain space of vector valued Siegel modular forms.

Known formula (Tsushima). For $w \leq 21$, dim 0 or 1, non zero iff :

$$(w, v) = \quad (17, 1) \quad (19, 7) \quad (21, 1) \quad (21, 5) \quad (21, 9) \quad (21, 13).$$

When $v \neq 1$, $\psi(\pi, \mathrm{St})$ cuspidal as $\mathrm{S}_{v+1}(\mathrm{SL}_2(\mathbb{Z})) = 0$.

When $v = 1$, might be $\Delta_w \oplus [2]$ .... and it is (Saito-Kurokawa form).

Example $G =$ definite SO_n for $n \leq 24$

$n \equiv 0 \bmod 8$,

$\mathcal{L}_n =$ set of even unimodular lattices in $\mathbb{R}^n$.

Choose $L \subset \mathbb{R}^n$.

$G = \mathrm{SO}_L$ semisimple over $\mathbb{Z}$, $G(\mathbb{R}) = \mathrm{SO}(\mathbb{R}^n)$, $\widehat{G} = \mathrm{SO}_n(\mathbb{C})$.

$G(\mathbb{Q})\backslash G(\mathbb{A})/G(\widehat{\mathbb{Z}}) = \mathcal{L}_n$

Number of π of G such that $\pi_\infty = 1$ (with mult.)

$= |\mathrm{SO}(\mathbb{R}^n)\backslash\mathcal{L}_n|$.

$= 1, 2, 25$ if $n = 8, 16, 24$ (Mordell, Witt, Niemeier).

Question: What are the $\psi(\pi, \mathrm{St})$?

Theorem (Ch.-Lannes) *They are :*

(i) $[15] \oplus [1]$ *and* $\Delta_{11}[4] \oplus [7] \oplus [1]$ *if* $n = 16$.

(ii) *the following if* $n = 24$ *:*

$[23] \oplus [1]$

$\mathrm{Sym}^2\Delta_{11} \oplus [21]$

$\Delta_{21}[2] \oplus [1] \oplus [19]$

$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{19}[2] \oplus [17]$

$\Delta_{21}[2] \oplus \Delta_{17}[2] \oplus [1] \oplus [15]$

$\Delta_{19}[4] \oplus [1] \oplus [15]$

$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{19}[2] \oplus \Delta_{15}[2] \oplus [13]$

$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{17}[4] \oplus [13]$

$\Delta_{17}[6] \oplus [1] \oplus [11]$

$\Delta_{21}[2] \oplus \Delta_{15}[4] \oplus [1] \oplus [11]$

$\Delta_{21,13}[2] \oplus \Delta_{17}[2] \oplus [1] \oplus [11]$

$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{19}[2] \oplus \Delta_{15}[2] \oplus \Delta_{11}[2] \oplus [9]$

$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{17}[4] \oplus \Delta_{11}[2] \oplus [9]$

$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{15}[6] \oplus [9]$

$\Delta_{15}[8] \oplus [1] \oplus [7]$

$\Delta_{21}[2] \oplus \Delta_{17}[2] \oplus \Delta_{11}[4] \oplus [1] \oplus [7]$

$\Delta_{19}[4] \oplus \Delta_{11}[4] \oplus [1] \oplus [7]$

$\Delta_{21,9}[2] \oplus \Delta_{15}[4] \oplus [1] \oplus [7]$

$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{19}[2] \oplus \Delta_{11}[6] \oplus [5]$

$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{19,7}[2] \oplus \Delta_{15}[2] \oplus \Delta_{11}[2] \oplus [5]$

$\Delta_{21}[2] \oplus \Delta_{11}[8] \oplus [1] \oplus [3]$

$\Delta_{21,5}[2] \oplus \Delta_{17}[2] \oplus \Delta_{11}[4] \oplus [1] \oplus [3]$

$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{11}[10] \oplus [1]$

$\Delta_{11}[12]$

Ikeda had found 20 of the 24 parameters (the ones without the $\Delta_{w,v}$), building on works of Nebe-Venkov, Freitag-Borcherds-Weissauer. Unconditional proof.

Case $n = 16$.

First check that π has a ϑ-correspondant on Sp_{2g} with $g < 8$. (Actually $g = 4$, as $\vartheta_g(\mathrm{E}_8 \oplus \mathrm{E}_8) \neq \vartheta_g(\mathrm{E}_{16})$ iff $g > 4$).

It shows $\psi(\pi, \mathrm{St})$ exists (Arthur, Rallis), say $\psi(\pi, \mathrm{St}) = \oplus_i \pi_i[d_i]$.

Inf. character : $\pm 7, \pm 6, \cdots, \pm 1, 0, 0$.

In part. $\mathrm{w}(\pi_i) + d_i - 1 \leq 14$ for each i, so $\pi_i = 1$ or Δ_{11} (Theorem).

Only two possibilites : the ones of the statements !

Happy Birthday Michael