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# NOTETAKER CHECKLIST FORM

**(Complete one for each talk.)**

**Name:** Pengcheng Xu **Email/Phone:** pengcheng.xu@okstate.edu

**Speaker's Name:** Ian Agol

**Talk Title:** The virtual Haken conjecture

**Date:** 03 / 18 / 2013 **Time:** 2 : 00 **am / pm (circle one)**

**List 6-12 key words for the talk:** Virtual Haken Conjecture; Thurston's question. virtual fibering question; cubulation

**Please summarize the lecture in 5 or fewer sentances:** The speaker presented the proof that cubulated hyperbolic group are virtually special. A consequence is that closed hyperbolic 3-manifolds have finite-sheeted Haken covers, which resolves the virtual Haken question of Waldhausen and Thurston's virtual fibering question.

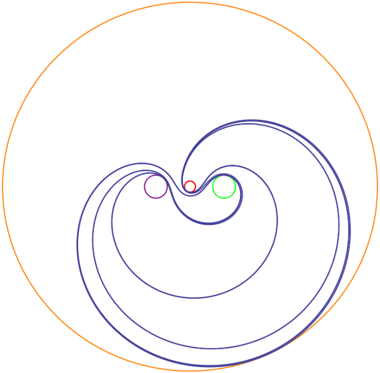
# CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

# The virtual Haken conjecture
## Hot Topics Workshop: Surface subgroups and cube complexes
## MSRI, March 18, 2013

Ian Agol



## Thurston's questions

In Thurston's paper *Three Dimensional Manifolds, Kleinian groups, and hyperbolic geometry* (BAMS '82), he asked 24 questions which have guided the last 30 years of research in the field. Four of the questions have to do with "virtual" properties of 3-manifolds:

- Question 15 (paraphrased): Are Kleinian groups LERF?
- Question 16: "Does every aspherical 3-manifold have a finite-sheeted cover which is Haken?" This question originated in a 1968 paper of Waldhausen.
- Question 17: "Does every aspherical 3-manifold have a finite-sheeted cover with positive first Betti number?"
- Question 18: "Does every hyperbolic 3-manifold have a finite-sheeted cover which fibers over the circle? This dubious-sounding question seems to have a definite chance for a positive answer."

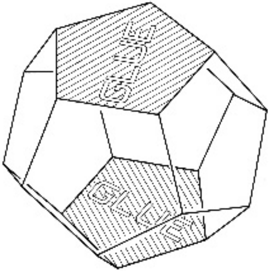
## Outline of the talk

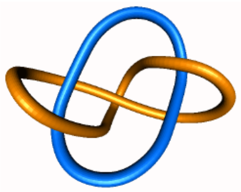
If a property holds for a finite sheeted cover of a manifold $M$, then we say that $M$ "virtually" has the property.
The goal of this talk will be to explain these questions about virtual properties of 3-manifolds, and how they reduce to a conjecture of Dani Wise in geometric group theory. If there's time, I'll describe something about the proof of this conjecture and thus the resolution of Thurston's questions.

With the geometrization theorem proved by Perelman '03 (Question 1 from Thurston's list), the most interesting case of questions 16-17 are for hyperbolic 3-manifolds, so we will focus for the most part on questions about virtual properties of hyperbolic 3-manifolds.
One remarkable feature of the proofs of these topological conjectures is that they almost entirely use geometric techniques.

## Some hyperbolic manifolds

*Hyperbolic 3-manifolds* admit a complete Riemannian metric of constant curvature $-1$, with fundamental group a *Kleinian group* (if it is finitely generated). Classic examples of hyperbolic 3-manifolds are the **Seifert-Weber dodecahedral space**, the **figure eight knot complement**, and the **Whitehead link complement**





### Definition

A subgroup $L < G$ is *separable* if for all $g \in G - L$, there exists $\phi : G \to K$, $K$ finite, such that $\phi(g) \notin \phi(L)$.

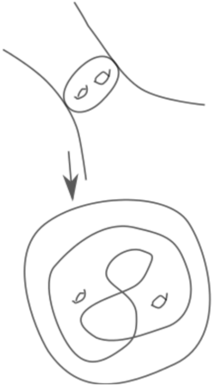
$G$ is *LERF* if all finitely generated subgroups are separable. $G$ is *QCERF* if $G$ is hyperbolic and quasiconvex subgroups are separable.

### Definition

A subgroup $L < G$ is *weakly separable* if for all $g \in G - L$, there exists $\phi : G \to K$ such that $\phi(L)$ is finite and $\phi(g) \notin \phi(L)$ ($K$ need not be finite).

## Locally Extended Residual Finiteness

Thurston's question 15 is whether Kleinian groups are LERF.
LERF allows one to lift $\pi_1$-injective immersions to embeddings in finite-sheeted covers (Scott '78)



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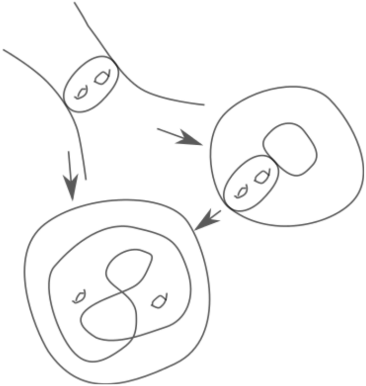


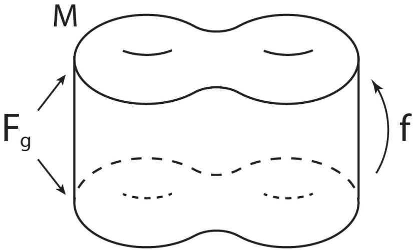
## Virtual Haken

- A compact 3-manifold $M$ (with hyperbolic interior) is **Haken** if it contains an embedded $\pi_1$-injective surface (e.g. a knot complement). The Seifert-Weber space is non-Haken (Burton- Rubinstein- Tillmann '12), as well as hyperbolic surgeries on the figure 8 knot complement (Thurston '78).
- A 3-manifold $M$ is **virtually Haken** if there is a finite-sheeted manifold cover $\tilde{M} \to M$ such that $\tilde{M}$ is Haken, e.g. hyperbolic surgeries on the figure 8 knot complement are virtually Haken (Dunfield-Thurston '03)
- Waldhausen '68 conjectured that every hyperbolic 3-manifold $M$ is virtually Haken (the *virtual Haken conjecture, Question 16).*
- A fortiori, does $M$ have a finite-sheeted cover $\tilde{M} \to M$ with $\pi_1(\tilde{M}) \twoheadrightarrow \mathbb{Z}$? (Question 17)

## Fibered 3-manifolds

A manifold $M$ **fibers over the circle** if there is a submersion $\eta : M \to S^1$. Each preimage $\eta^{-1}(x)$ is a codimension-one submanifold of $M$ called a **fiber**. If $M$ is 3-dimensional and fibers over $S^1$, then the fiber is a genus $g$ surface $F_g$, and $M$ is obtained as the mapping torus of a homeomorphism $f : F_g \to F_g$,

$$M \cong T_f = \frac{F_g \times [0,1]}{\{(x,0) \sim (\phi(x),1)\}}.$$



## Virtually fibered 3-manifolds

- $M$ is **virtually fibered** if there exists a finite-sheeted cover $\tilde{M} \to M$ such that $\tilde{M}$ fibers
- If $M$ fibers, then $b_1(M) > 0$, so this is stronger than asking for virtual positive betti number.
- There have been several classes of hyperbolic 3-manifolds shown to virtually fiber, including 2-bridge links, some Montesinos links, and certain alternating links (Agol-Boyer-Zhang, Aitchison-Rubinstein, Bergeron, Chesebro-DeBlois-Wilton, Gabai, Leininger, Reid, Walsh, Wise).
- Thurston asked whether every hyperbolic 3-manifold is virtually fibered (Question 18)?

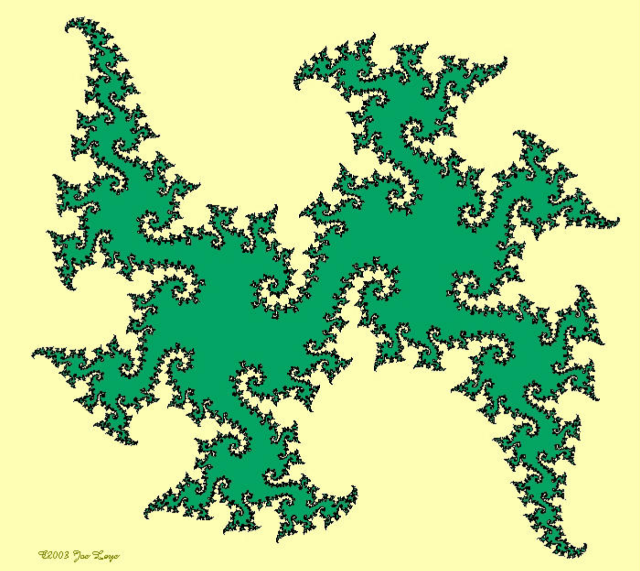
## Surfaces in hyperbolic 3-manifolds

If $M$ is a finite volume hyperbolic 3-manifold, and $f : \Sigma_g \to M$ is an essential immersion of a surface of genus $g > 0$, then there is a dichotomy for the geometric structure of the surface discovered by Thurston, and proven by Bonahon in general. Either $f$ is

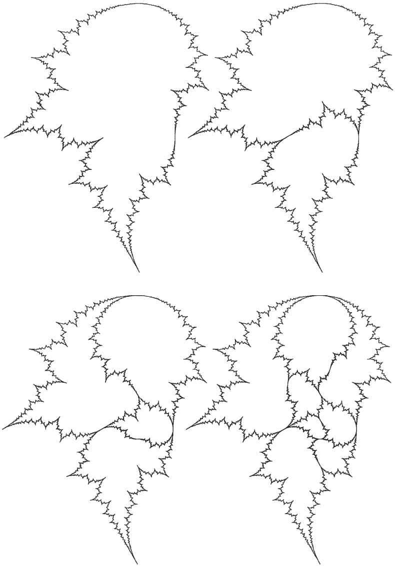
- **geometrically finite** or
- **geometrically infinite**.

The first case includes **quasifuchsian** surfaces. In the geometrically infinite case, the surface is **virtually the fiber** of a fibering of a finite-sheeted cover of $M$, and the subgroup $f_\#(\pi_1(\Sigma_g)) < \pi_1(M)$ is separable. The **Tameness theorem** (A., Calegari-Gabai) plus the **covering theorem** of Canary implies a similar dichotomy for finitely generated subgroups of $\pi_1(M)$: either a subgroup is geometrically finite, or it corresponds to a virtual fiber. This result is used in proving that certain Kleinian groups are LERF, since the subgroups corresponding to virtual fibers are separable.

## Quasi-fuchsian surface group limit set



# Part of the Peano curve "limit set" of the figure eight fiber



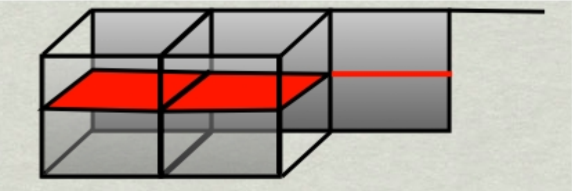
## Cubulations

A topological space $Y$ is **cubulated** if it is *homotopy equivalent* to a compact locally CAT(0) cube complex $X \simeq Y$. We will be interested in 3-manifolds which are cubulated.
**Remark:** If $X \simeq M^3$ is a CAT(0) cubing of a closed 3-manifold, then *dim*$X$ may be $> 3$. Tao Li '02 has shown that there are hyperbolic 3-manifolds $M$ such that there is no **homeomorphic** CAT(0) cubing $X \cong M$.

Theorems of Sageev '94 associate a cocompact action of $\pi_1(M)$ on a (globally) CAT(0) cube complex if $M$ contains a $\pi_1$-injective surface. Globally CAT(0) $\iff$ simply-connected and locally CAT(0)

## Cubulations

Sageev's construction gives a cube complex in which each immersed essential surface in a 3-manifold corresponds to an orbit of an embedded hyperplane.



For example, in the case of a surface which is virtually a fiber, Sageev's construction gives rise to a crystallographic group action and an embedded essential surface gives an action on a tree.

## Essential surfaces in hyperbolic 3-manifolds

**Theorem (Kahn-Markovic 2009)**

*Hyperbolic 3-manifolds contain immersed quasi-fuchsian surfaces which are arbitrarily close to being totally geodesic.*

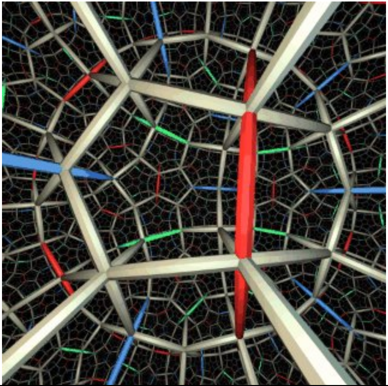
**Theorem (Bergeron-Wise 2009)**

*Closed hyperbolic 3-manifolds are cubulated.*

These theorems will be discussed in Jeremy Kahn's talk.

## Cubulated hyperbolic 3-manifolds

There were many known examples of cubulated hyperbolic 3-manifolds before this theorem, e.g. alternating link complements [Aitchison-Rubinstein '92]. Other examples come from tessellations by right-angled polyhedra and arithmetic 3-manifolds (Lackenby '08):



## Previous theorem on virtual fibering

**Theorem (A. 2008)**

*If $M^3$ is virtually special cubulated, then $M$ is virtually fibered.*

Since $M$ is cubulated, $M \simeq X$, where $X$ is a CAT(0) compact cube complex.
There is a finite-sheeted cover $\tilde{X}$ which is **special**, and implies that $\pi_1(\tilde{X}) < RAAG$. A strong form of residual solvability for RAAG's called **RFRS** passes to $\pi_1(\tilde{X})$ and implies that $M$ is virtually fibered (A.).

## Theorem on virtual fibering

**Theorem (Wise 2011)**

*Virtually quasi-fuchsian Haken hyperbolic 3-manifolds are virtually special cubulated, and therefore Haken hyperbolic 3-manifolds are virtually fibered.*

Wise's theorem gives a different approach to finding cubulations (based on work with Hsu) than the result of Bergeron-Wise, and holds in much greater generality than stated here.
As indicated in Grove's talk, this holds for hyperbolic groups which admit a quasiconvex hierarchy.

## Wise's conjecture on virtually special cubulations

Wise conjectured the following in 2011:

**Theorem (A. 2012)**

*Cubulations with hyperbolic fundamental group are virtually special.*

**Corollary**

*Let $M$ be a closed hyperbolic 3-manifold. Then $\pi_1 M$ is LERF and $M$ virtually fibers.*

This resolves positively Thurston's questions 15-18. The proof of this theorem makes use of Wise's results, in particular the *Malnormal Special Quotient Theorem*.

## Wise's conjecture

Part of the argument is based on joint work with Groves and Manning:

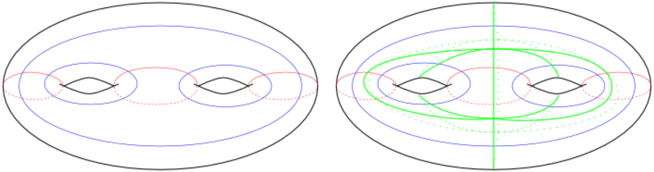
**Theorem (A.-Groves-Manning 2012)**

*Let $G$ be a hyperbolic group, and let $H < G$ be a quasiconvex virtually special subgroup. Then $H$ is weakly separable in $G$.*

The proof of this makes use of *hyperbolic Dehn filling*, as well as an inductive method of "height reduction" for quasiconvex subgroups developed in our previous work and the MSQT.

## Outline of the proof

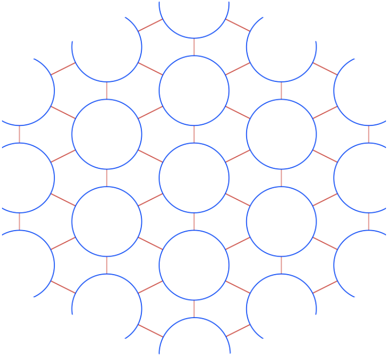
We'll discuss the proof in the context of a toy model. Consider a compact hyperbolic surface $X$ with an immersed geodesic curve:



In fact, in this picture, the curves are dual to a CAT(0) square complex. The first step is to construct an infinite sheeted regular cover $\mathcal{X}$ with the elevations of the curves compact by applying weak separability (in the surface case, this step does not need the MSQT, just hyperbolic Dehn filling, or rather its ancestor small-cancellation theory a la Gromov).

## Cover with compact walls

In our particular case, kill the red curves, and the third power of the blue curves to obtain $\mathcal{X}$:



## Outline of proof

### Definition (Crossing Graph)

Let $\Gamma(\mathcal{X})$ be a graph with vertex set $V(\Gamma(\mathcal{X})) = \mathcal{W}$ the hyperplanes of $\mathcal{X}$, and edges $(W_1, W_2) \in E(\Gamma(\mathcal{X}))$ if $W_1 \cap W_2 \neq \emptyset$ or if there is an essential cylinder going between $W_1$ and $W_2$.

### Definition (Coloring space)

Let $[n] = \{1, \ldots, n\}$. Let

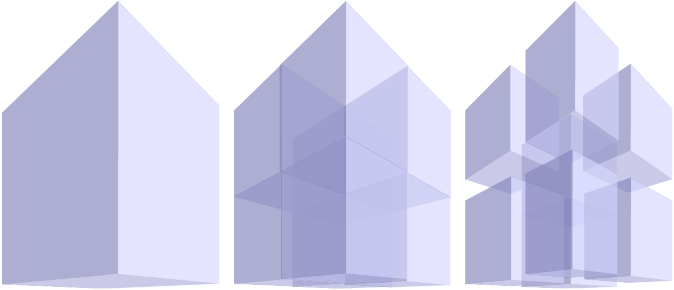
$$C_n(\Gamma) = \{c : V(\Gamma) \to [n] | c(W_1) \neq c(W_2), \forall (W_1, W_2) \in E(\Gamma)\}$$

denote the space of $n$-colorings of the graph $\Gamma$.

We regard $C_n(\Gamma)$ as a closed subspace of the Cantor set $[n]^{V(\Gamma)}$. If $deg(\Gamma) \leq k$, then $C_{k+1}(\Gamma) \neq \emptyset$.

## Colorings and hierarchies

A coloring $c \in C_n(\Gamma(\mathcal{X}))$ gives rise to a hierarchy of $\mathcal{X}$: cut along the walls colored 1, then the walls colored 2, ..., and finally the walls colored $n$.



What is left at the ends are stars of the vertices of $\mathcal{X}$, with residues of the colorings remaining on the boundaries.

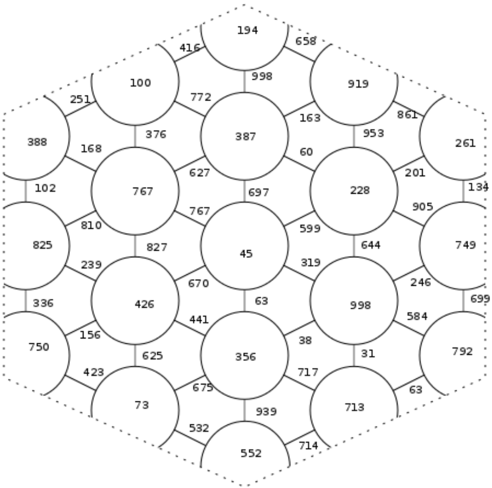
## Colorings of graphs

### Lemma

*There exists a probability measure $\mu$ on $C_{k+1}(\Gamma)$ which is $\mathcal{G}$-invariant.*

The proof of this lemma proceeds by coloring the vertices $V(\Gamma)$ randomly with $n$-colors, $n \geq k+1$. The probability that two endpoints of and edge $e \in E(\Gamma)$ have the same color is $1/n$. One can produce an $n-1$-coloring of the vertices, by sending each vertex colored $n$ each color to the smallest color unused by its neighbors. By induction then, one produces a measure on $k+1$-colorings of $V(\Gamma)$ which have probability of coloring the endpoints of $e$ the same color as $1/n$. Taking a weak-* limit of these measures, one obtains a $\mathcal{G} - invariant$ measure $\mu$ on $V(\Gamma)^{[k+1]}$ which is supported on the colorings of $\Gamma$.

# A random 1000 coloring



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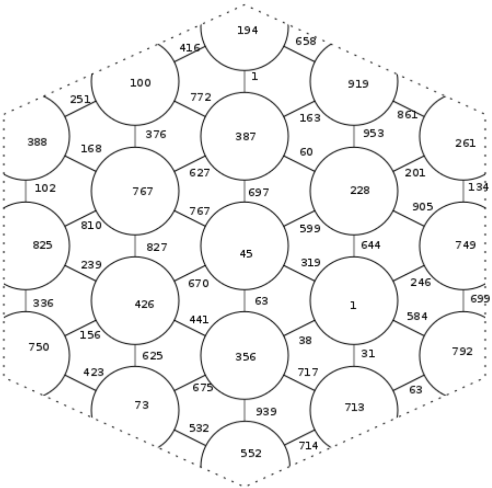

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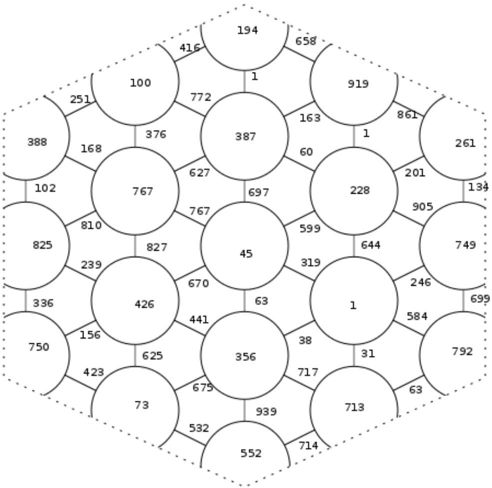

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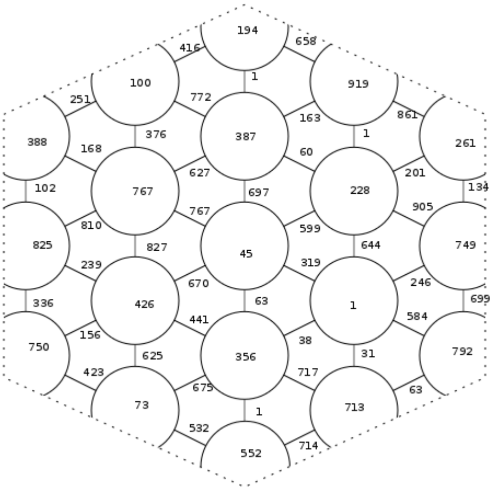

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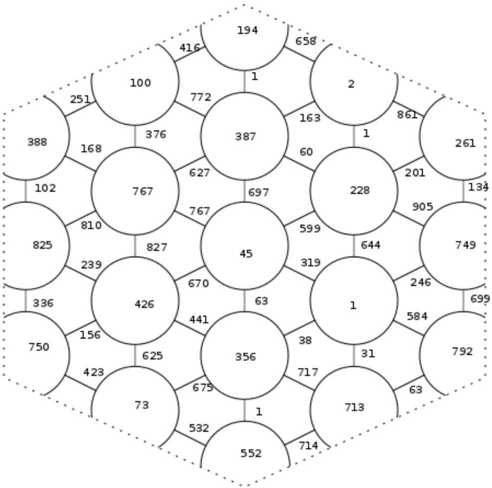

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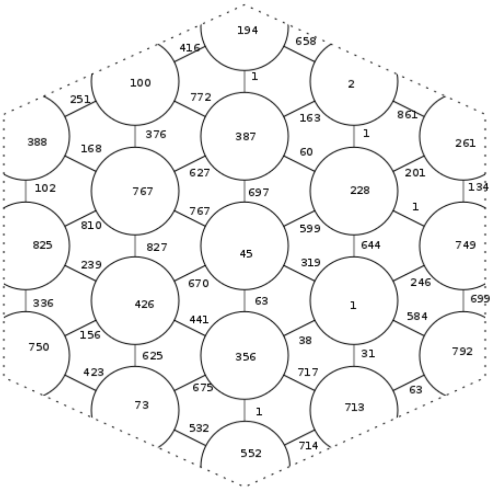

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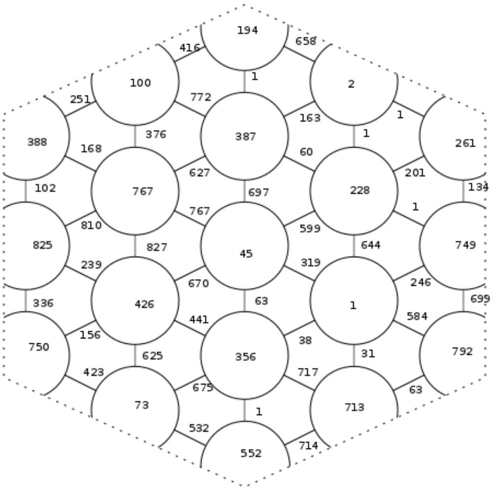

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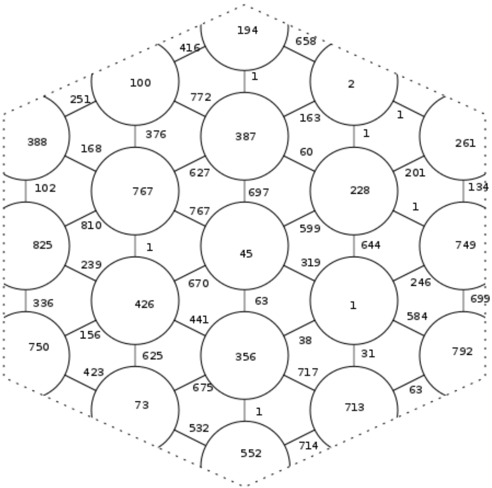

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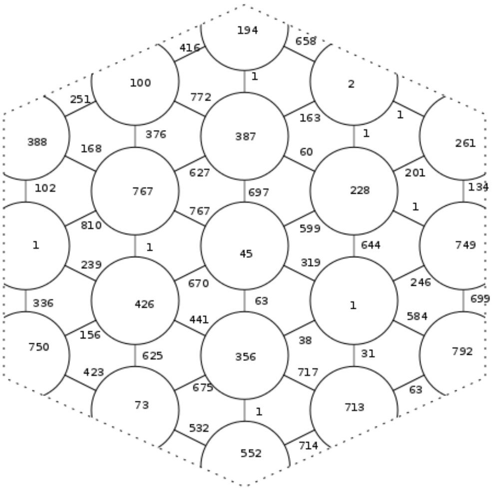

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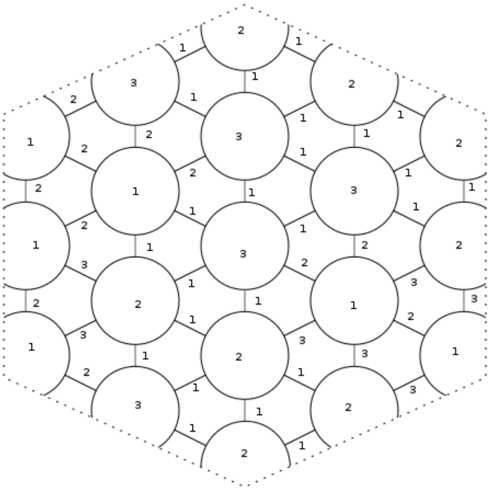
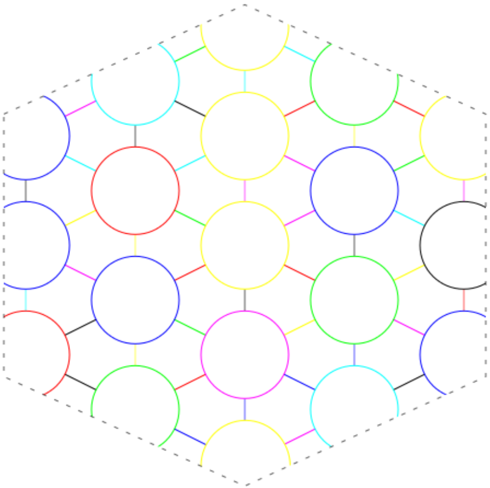


# Coloring Γ with 7 colors

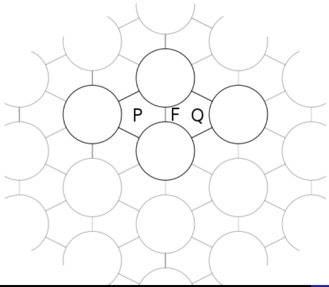


## Colorings and hierarchies

The probability measure is just an artifice to construct a solution to the *gluing equations*. We want to reverse engineer a hierarchy of a finite-sheeted cover. We have an finite (non-compact) hierarchy associated to the cover $\mathcal{X}$. The probability measure allows us to extract some finiteness associated to this hierarchy.

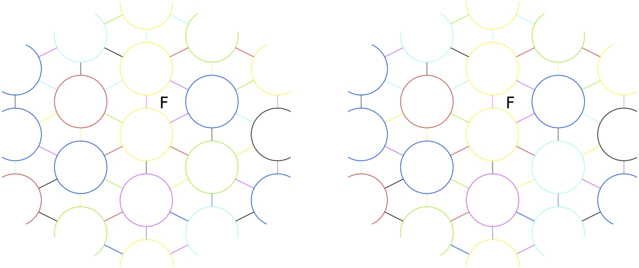
## Polyhedra and facets

Let $\mathcal{P}$ denote the stars of vertices of $\mathcal{X}$, which we will call *polyhedra*. Let $\mathcal{F}$ denote the facets of $\mathcal{X}$, which are dual to each edge of $\mathcal{X}$, and are the facets of the polyhedra $\mathcal{P}$. Each facet $F \in \mathcal{F}$ will be contained uniquely in two polyhedra $P, Q \in \mathcal{P}$, $P \cap Q = F$. There are 4 polygons in the example up to the action of $\mathcal{G}$ (we won't draw $P'$ and $Q'$ which are duplicates of $P$ and $Q$).



## Supercoloring

Each polyhedron and facet of $\mathcal{X}$ will correspond uniquely to one of $X$ via the covering $\mathcal{X} \to X$. We refine the $k+1$-coloring of the walls $\mathcal{W}$ by the coloring of a neighborhood of size $j$, where $j$ is the color of a vertex. The facets $F \in \mathcal{F}$ get super colored by their corresponding walls, and polyhedra will be super colored by their facets.



## Polyhedral gluing equations

The variables for the gluing equations will be super colored polyhedra, and the gluing equations will say that for a given super colored facet $F$, the super colorings of $P$ which induce the same super coloring of $F$ must equal the super colorings of $Q$ which induce the super coloring of $F$. We require that the variables are $\mathcal{G}$-invariant, in which case they are determined by finitely many variables corresponding to the polyhedra of $X$ (or $\mathcal{G}$-orbits of super colored polyhedra). The $\mathcal{G}$-invariant measure $\mu$ gives us a solution to the gluing equations with non-negative weights. Then we can get an integral solution to the gluing equations with non-negative weights, since the equations are linear equations with integral coefficients.

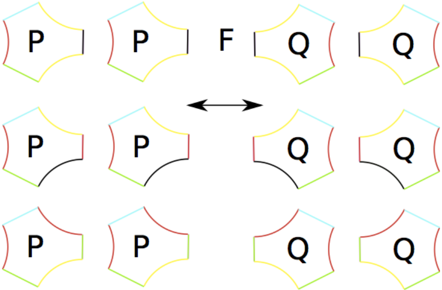
We take the integral solution to the polyhedral gluing equations, and use them to glue up a finite-sheeted cover of $X$, which is "modeled" on the hierarchies associated to colorings of $\mathcal{X}$.

## Gluing up the herarchy

We construct a sequence of (usually disconnected) finite cube complexes $\mathcal{V}_j$, $k+1 \geq j \geq 0$, with boundary pattern $\{\partial_1(\mathcal{V}_j), \ldots, \partial_j(\mathcal{V}_j)\}$ determined by the unpaired faces colored $j$.
The final stage $\mathcal{V}_0$ will be a finite-sheeted cover of $X$.
The first stage $\mathcal{V}_{k+1}$ is obtained by taking a number of copies of each supercolored polyhedron determined by the integral solution to the gluing equations.
If we glued the faces of the polyhedra $\mathcal{V}_{k+1}$ together preserving colors, then we would obtain a finite-sheeted branched cover of $X$.
So we have to be careful at each stage that the gluing extends to an unbranched covering space.

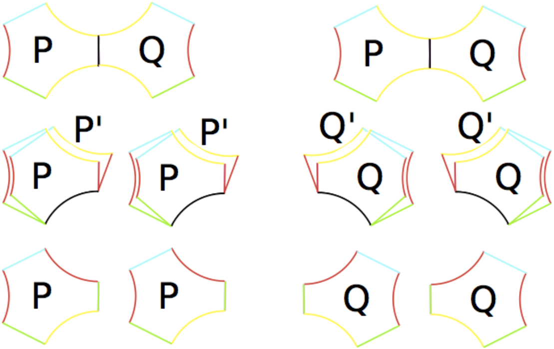
## Gluing polyhedra satisfying the gluing equations

The super coloring guarantees that when we glue together super colored polyhedra $P$ and $Q$ along the face $F$ so that they satisfy the gluing equations, then the resulting boundary pattern will be super colored in a consistent fashion, allowing a hierarchy to be constructed inductively.



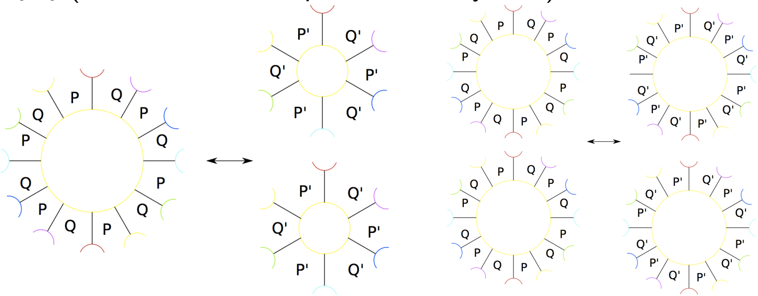
## Gluing up the hierarchy

The base case $\mathcal{V}_{k+1}$ is the collection of equivalence classes of polyhedra given by the solution to the polyhedral gluing equations. $\mathcal{V}_k$ is obtained from $\mathcal{V}_{k+1}$ by gluing the faces labeled $k+1$ in pairs along matching supercolored faces (in our example, $k+1=7$ is represented by black, obtaining $\mathcal{V}_6$).



## Gluing up the hierarchy

We glue $\mathcal{V}_5$ from a cover of $\mathcal{V}_6$ by gluing the boundary pattern $\partial_6\mathcal{V}_6$ (which in our example is colored yellow):



The supercoloring guarantees that the two sides of $\partial_6\mathcal{V}_6$ have consistently supercolored walls, and therefore is a finite-sheeted cover of the wall in a representative coloring of $\mathcal{X}$. The MSQT allows us to pass to a finite-sheeted cover $\tilde{\mathcal{V}}_6$ in which both sides of $\partial_6\tilde{\mathcal{V}}_6$ match.

## Gluing up the hierarchy

We obtain $\mathcal{V}_i$ from $\mathcal{V}_{i+1}$ by finding a covering space $\tilde{\mathcal{V}}_{i+1} \to \mathcal{V}_{i+1}$ in which the boundary pattern $\partial_{i+1}\tilde{\mathcal{V}}_{i+1}$ may be matched up in pairs which reverse the coorientations and preserve super colorings. Constructing this cover requires another application of Wise's MSQT.

The cube complex $\mathcal{V}_0$ will have no boundary pattern, and thus will give a finite-sheeted covering space $\mathcal{V}_0 \to X$ and which has by construction has embedded walls, and therefore a malnormal hierarchy.

One more application of Wise's theorem (rather Haglund-Wise) gives a cover $\tilde{V}_0 \to X$ which is special.

# The End

In memory of Bill 1946-2012