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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Pengcheng Xu **Email/Phone:** pengcheng.xu@okstate.edu

Speaker's Name: Danny Calegari

Talk Title: Surface subgroups in random graphs of free groups

Date: 03 / 19 / 2013 **Time:** 11 : 00 am / pm (circle one)

List 6-12 key words for the talk: surface subgroup; free group; random group; cubulation; Gromov's question; scl

Please summarize the lecture in 5 or fewer sentances: The speaker started with Gromov's question, then discussed the theorem that a random graph of free groups contains a surface subgroup. He continued to discuss the construction of fatgraphs and linear programming.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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surface subgroups in random groups

Danny Calegari

March 19, 2013

1. Gromov's Surface Subgroup Question

The following is "Gromov's Surface Subgroup Question":

Question: Let G be a one-ended hyperbolic group. Does G contain a subgroup isomorphic to $\pi_1(S)$ where S is a closed surface with $\chi(S) < 0$?

Part of a broader set of questions:

1. What "interesting" subgroups does G have?
2. How do you show that a map $\pi_1(S) \to G$ is injective?
3. Is there a "good" surface subgroup? (e.g. quasiconvex, norm minimizing in a homology class)

Surfaces are a "bridge" from hyperbolic geometry to symplectic geometry (causal structures, quasimorphisms, scl, etc.)

Strategy: find an intermediate class of groups $\mathcal{G}$ so that

1. every one-ended hyperbolic group contains a subgroup in $\mathcal{G}$;
2. every group in $\mathcal{G}$ contains a surface subgroup.

Proposal 1: $\mathcal{G}$ is the class of one-ended graphs of free groups.

Proposal 2: $\mathcal{G}$ is the class of cubulated groups.

Proposition: Every one-ended cubulated hyperbolic group contains a one-ended graph of free groups (uses Agol's Theorem).

Previous work: surface subgroups in certain graphs of free groups.

1. C.-: cyclic edge groups and $b_2 > 0$
2. Kim, Kim-Oum, Kim-Wilton: doubles along cyclic edge groups
3. C.-Walker: doubles along high rank subgroups and $b_2 > 0$

Definition: G is a random group at density D if it is obtained from a free group F_k of rank $k \geq 2$ by adding $(2k-1)^{nD}$ random relations of length n.

Theorem (Gromov):

1. If $D > 1/2$ then G is trivial or $\mathbb{Z}/2\mathbb{Z}$.
2. If $D < 1/2$ then G is infinite hyperbolic and one-ended.
3. G is $C'(2D)$.

Proposal 3: $\mathcal{G}$ is the class of random groups (at $D < 1/2$).

Theorem (Wise): A $C'(1/6)$ group is cubulated.

Theorem (Ollivier-Wise): A random group at $D < 1/6$ is cubulated.

Theorem (Zuk): A random group at $D > 1/3$ has property (T) and is therefore NOT cubulated.

2. Statement of Theorems

Definition: Fix F_k, F_l free groups of ranks k, l with fixed generating sets. A random homomorphism of length n is a homomorphism

$$\phi : F_k \to F_l$$

sending generators of F_k to random (reduced) words in F_l of length n.

Theorem (C.-Walker): A random ascending HNN extension of a free group (of rank ≥ 2) contains a surface subgroup.

Theorem (C.-Wilton): A random HNN extension or amalgamated free product of free groups with edge groups of rank ≥ 1 contains a surface subgroup.

Hence a random graph of free groups contains a surface subgroup.

A random group at density $D < 1/6$ is cubulated, and therefore contains a (one-ended) graph of free groups.

Question: Is this a random graph of free groups?

Theorem (C.-Walker): A random group at sufficiently low density D contains a surface subgroup.

The argument probably works for $D < 1/12$ (so that G is $C'(1/6)$). But maybe the technique can be pushed higher (to $D < 1/2$?)

The argument is constructive. One can actually build the surface and (in principle) see that it is injective. For the random graph of free groups this is effective in practice.

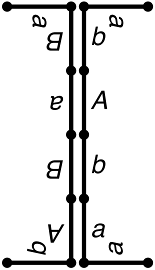
The surfaces are quasiconvex. And there should be many of them (g^{Cg} of genus $g \gg 1$).

The surfaces are constructed by a mix of combinatorics and ergodic theory. The main technical step is to build an injective surface with prescribed boundary in a free group.

In the language of fatgraphs: given a collection of cyclic words g_i in F_k, we look for a folded fatgraph Y such that $\partial S(Y) = \cup_i g_i$.

Theorem (C.-Walker): If the g_i are random, such a Y exists.

We build the fatgraph Y by finding pairs of segments σ, σ^{-1} with opposite labels in $\cup_i g_i$, and pairing them. When all of $\cup_i g_i$ is paired in this way, we will have built a fatgraph.



3. Constructing the fatgraphs

Suppose w is a long random reduced word of length n. There are $4 \times 3^{m-1}$ reduced words of length m, for any m, so if v is any reduced word of length m, the expected number of copies of v in w is $\frac{3}{4}n/3^m$.

Chernoff's inequality says that for $m < (1-\epsilon)\log_3 n$, for every reduced word v of length m, there is an inequality

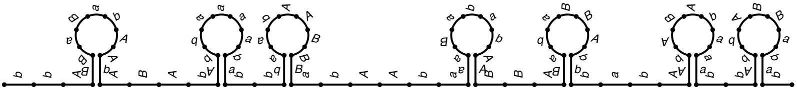
$$1 - \epsilon \leq \frac{\#\text{ of copies of } v \text{ in } w}{\text{expected }\#\text{ of copies of } v \text{ in } w} \leq 1 + \epsilon$$

with probability at least $1 - O(e^{-n^c})$ (when n is big).

In other words, there is equidistribution at scales below $\log_3 n$ with an error that goes to zero exponentially fast.

Step 1: Fold off short loops.

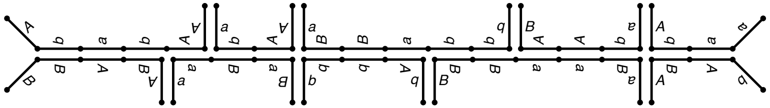
As we read along the word w, look for strings of length 11 of the form $yxuXz$ for inverse letters x, X, non-inverse letters y, z and $|u| = 7$. When we see such a string, we glue x to X and fold off a short loop. Each short loop is determined by the word u and the letter x.



The result of this is to replace a word w of length n by a new word w' of length $(1 - \frac{9}{7}\alpha)n$ and a reservoir $\mathcal{L}$ of short loops, of total length αn.

Step 2: Cancel most of w'.

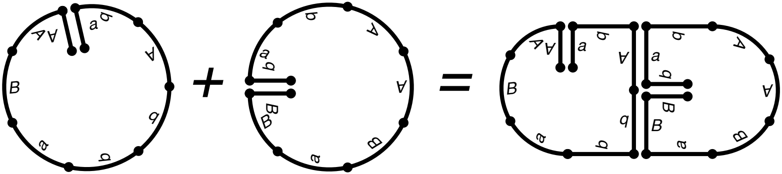
Partition w' into a sequence of consecutive substrings of length ϵ^{-1} (with a small fixed bit of space between). Match copies of substrings of the form σ with copies of the inverse substring σ^{-1}. By Chernoff, we can glue up almost all of w' this way.



The result of this is to replace w' of length $(1 - \frac{9}{7}\alpha)n$ with a new collection w'' of loops of total length ϵn. Let's assume $\alpha \gg \epsilon$.

Step 3: Build W'' from the reservoir.

If we have enough short loops of every kind, we can glue them together to build a copy of W'', the inverse of w''. This can be paired with w''.



The result of this is to cancel w'' at the cost of slightly adjusting the inventory in the reservoir. The new reservoir $\mathcal{L}'$ of short loops still has total length of order αn.

Step 4: Glue up $\mathcal{L}'$.

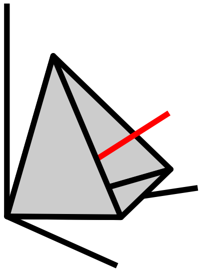
By Chernoff, the initial distribution of short loops in $\mathcal{L}$ was very close to uniform. After using up ϵn short loops to cancel w'', we still have a very nearly uniform distribution of short loops in $\mathcal{L}'$ of total length αn. So all that remains to show is:

Claim: Let $\mathcal{L}'$ be a collection of short loops which is sufficiently close to uniformly distributed. Then $\mathcal{L}'$ bounds a folded fatgraph.

4. Linear Programming

Let V be the vector space spanned by the set of short loop types, and let V^+ be the positive cone. There is a feasible subcone C of V^+ consisting of formal linear combinations of short loops that bound a formal linear combination of folded fatgraphs.

Let Δ denote the simplex obtained by projectivizing V^+, and let $\mathbb{P}C \subset \Delta$ be the image of C. We need to show that the uniform vector $\mathbf{1}$ projects into the interior of $\mathbb{P}C$.



Actually, in principle, the cone C can be computed directly by linear programming. However, this computation is not possible in practice.

There are 4376 short loops of length 7, and there are roughly 3×10^{15} different kinds of piece in the folded fatgraphs they bound.

Instead, we look for sparse vectors in C — those with very few nonzero coordinates. A big enough collection of these vectors span a polyhedron in $\mathbb{P}C$ with $\mathbf{1}$ in the interior.

References

- D. Calegari and A. Walker, *Surfaces from linear programming*. preprint, arXiv:1212.2618
- D. Calegari and H. Wilton, *Random graphs of free groups contain surface subgroups*. preprint, arXiv:1303.2700
- D. Calegari and A. Walker, in preparation