MSRI

Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Pengcheng Xu Email/Phone: pengcheng.xu@okstate.edu

Speaker's Name: Alan Reid

Talk Title: Recognizing free groups, surface groups and Kleinian groups by their finite quotients

Date: 03 / 19 / 2013 Time: 3 : 30 am / pm (circle one)

List 6-12 key words for the talk: residually finite group; profinite completion; ~~first~~ 1^{st} L^2-betti number; good group

Please summarize the lecture in 5 or fewer sentances: The speaker talked about the recent work aimed at distinguishing finitely presented residually finite groups by their finite quotients.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Tuesday 4th talk | Recognizing free groups, surface groups and Kleinian groups by their finite quotients.

speaker: Alan Reid.

joint with M. Budson, M. Budson ~~+~~ M. conder

All groups are f.g (f.p) and Residually Finite (RF)

Def: Γ is RF if $\forall g \in \Gamma$ $(\neq 1)$, $\exists \varphi: \Gamma \to A$ $|A| < \infty$ with $\varphi(g) \neq 1$

Def: Γ as above, $\mathcal{C}(\Gamma) = \{G : G \text{ is a finite quotient of } \Gamma\}$
(isomorphism classes of finite quotients of Γ)

Def: $\mathcal{G}(\Gamma) = \{\Delta : \mathcal{C}(\Gamma) = \mathcal{C}(\Delta)\}$
genus of Γ

Natural questions: ① What are examples of Γ with $\mathcal{G}(\Gamma) = \{\Gamma\}$.
or $|\mathcal{G}(\Gamma)| > 1$

② What φ ? are shared for groups in same genus?

③ different

④ How big can $|\mathcal{G}(\Gamma)|$ be?

Additional Notation: P is some class of groups.

denote $\mathcal{G}(\Gamma, P) = \{\Delta : \mathcal{C}(\Gamma) = \mathcal{C}(\Delta), \Delta \in P\}$.

Example: 1. $\mathcal{G}(\mathbb{Z}) = \{\mathbb{Z}\}$

Proof: $\Gamma \in \mathcal{G}(\mathbb{Z})$. If Γ is not abelian, then $\exists\, c = aba^{-1}b^{-1} \neq 1$

Γ RF $\Rightarrow \exists \varphi: \Gamma \to A$ $|A| < \infty$ $\varphi(c) \neq 1$

Commutators die in cyclic quotients.

Observation: If $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2) \Rightarrow b_1(\Gamma_1) = b_1(\Gamma_2)$

Emphasis of today's talk is in free groups, surface group + Kleinian groups

Open question (70's) Is $\mathcal{G}(F_r) = \{F_r\}$ $\overset{\text{rank}}{r} \geq 2$

Example: $\mathcal{G}(F_r)$ doesn't contain a surface group of genus g.

$F_r = F_{2g}$ free on $2g$-genus by observation.

Consider $F_{2g} \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z}$

$$\begin{array}{l} \quad\quad\quad\quad \triangledown \\ \left.\begin{array}{l}\text{free of rank} \\ 4g-1\end{array}\right) \; K = \operatorname{Ker}\varphi \longrightarrow H_1(K, \mathbb{F}_p) \quad p \text{ prime} \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \cong (\mathbb{Z}/p\mathbb{Z})^{4g-1} \\ \quad\quad\quad\quad | \\ \quad\quad H_p = \operatorname{Ker}\varphi_p \end{array}$$

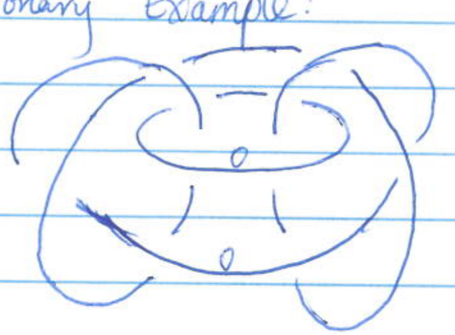
$H_p \trianglelefteq F_{2g}$ and $F_{2g}/H_p = Q$ $\quad |Q| < \infty$

Q is a quotient of $\pi_1(\Sigma g)$

But an index 2 subgroup of $\pi_1 \Sigma g$ cannot surject $(\mathbb{Z}/p\mathbb{Z})^{4g-1}$

Cautionary Example:

$(\frac{0}{1}, \frac{0}{1})$ surgery gives $(S^2 \times S^1) \# (S^2 \times S^1)$ $\rightarrow \Gamma \twoheadrightarrow F_2$.



L

$\Gamma = \pi_1(S^3 \setminus L)$

(hyperbolic)

cannot distinguish Γ from F_2 by

- finite simple quotient
- finite nilpotent quotient

Stallings: Γ and F_2 have same LCS quotients.

Free groups
- $\rightarrow$ f.g. Fuchsian groups
- $\rightarrow$ lattices in connected Lie group. $(\mathcal{L})$
- $\rightarrow$ hyperbolic groups $(\mathcal{H})$
- $\rightarrow$ fully residually free groups $(\mathcal{F})$
- $\rightarrow$ Right-Angled Artin groups (RAAG's) $(\mathcal{R})$
- $\dashrightarrow$ parafree groups

Thm 1 (B-C-R) Let $\Gamma = F_r$ $(r \geq 2)$. Then

(1) $\mathcal{G}(\Gamma, \mathcal{L}) = \{\Gamma\}$ (replace Γ by f.g. Fuchsian group)

(2). $\mathcal{G}(\Gamma, R) = \{\Gamma\}$

(3) If $\Delta \in \mathcal{G}(\Gamma, \mathcal{F})$, then Δ is hyperbolic and does not contain a surface group.
($\Delta \neq$ free)

(4). If $\Delta \in \mathcal{G}(\Gamma, \mathcal{H})$ is one-ended, then either $\exists$ a hyperbolic group that is NOT RF or $\exists$ a one ended hyperbolic group that ~~does not~~ does not contain a QC surface subgroup.

To what extent does $\mathcal{C}(\overset{T^2}{\pi}, M^3)$ (M^3 compact) determine $\pi_1(M^3)$

Ex (Funer) $\exists M_1, M_2$, torus boundles over S^1 (with SOL geometry) s.t. $\mathcal{C}(\pi, M_1) = \mathcal{C}(\pi, M_2)$ $M_1 \not\cong M_2$

Thm 2 (Consequence of Agol, Wise) Let $M_1 = \mathbb{H}^3/\Gamma_1$ be closed, $M_2 = \mathbb{H}^3/\Gamma_2$ be non-compact finite volume. Then $\mathcal{C}(\Gamma_1) \neq \mathcal{C}(\Gamma_2)$

Remark: M.Aka - constructs lattices Γ in higher rank with $|\mathcal{G}(\Gamma)| > n$

Organizing finite quotients: The profinite completion of Γ

Let $\mathcal{N} = \{$ normal subgroups of finite index in $\Gamma \}$

$\hat{\Gamma} = \varprojlim \{\Gamma/N : N \in \mathcal{N}\}$.

Think more concretely:

$\Gamma \xrightarrow{i} \prod \{\Gamma/N : N \in \mathcal{N}\}$ $\quad$ (Γ is RF $\Rightarrow$ i is 1-1)

$g \longrightarrow (gN)$

$\hat{\Gamma} = \overline{i(\Gamma)} \subset \prod$

Correspondance Thm:

$\{$finite index subgroups in $\Gamma\} \longleftrightarrow \{$finite index subgroups in $\widehat{\Gamma}\}$

Key Thm: $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2) \Leftrightarrow \widehat{\Gamma}_1 \cong \widehat{F}_2$

Remark: In Thm 1, the hypothesis gives

discrete groups $\xhookrightarrow{1-1} \widehat{\Gamma}$ (Γ is free)

In Thm 2, we have $\mathbb{Z} \oplus \mathbb{Z} \to \widehat{\Gamma}_1$

Proof of Thm 1, 2. In Thm 1, (1).

Proposition: If $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2 \Rightarrow b_1^{(2)}(\Gamma_1) = b_1^{(2)}(\Gamma_2)$

Luck Approximation Thm

Let $\{N_i\}$ be finite index normal subgroups of Γ

$\Gamma > N_1 > \dots > N_i > \dots$

$$b_1^{(2)}(\Gamma) = \lim_{n\to\infty} \frac{b_1(N_i)}{[\Gamma : N_i]}$$

Thm 1(4). $\Delta \in \mathcal{G}(\Gamma)$ with Δ contains a QC surface subgroup Σ.

$\Sigma \hookrightarrow \Delta \hookrightarrow \widehat{\Delta} \cong \widehat{F}$

By Agol-Groves-Manning $\Rightarrow \Sigma$ and all its finite index subgroups are separable in Δ

$\Rightarrow \overline{\Sigma} \subset \widehat{\Delta}$

$\begin{array}{c} \cong \\ \widehat{\Sigma} \subseteq \widehat{F} \end{array}$

closed

Then Serre gives cd (closed subgroups) ≤ 1

cohomological-dimension