MSRI
Mathematical Sciences Research Institute

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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Pengcheng Xu Email/Phone: pengcheng.xu@okstate.edu

Speaker's Name: Mladen Bestvina

Talk Title: scl in mcg

Date: 03/20/2013 Time: 11:00 am / pm (circle one)

List 6-12 key words for the talk: stable commutator length; mapping class group; quasi-morphism; quasi tree

Please summarize the lecture in 5 or fewer sentances: The speaker discussed the following work: Given an element f of the mapping class group of a compact surface, they can read off from its Nielsen-Thurston decomposition whether or not scl(f)>0, and there is a similar algorithm for subgroups of mcg.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
    - <u>**Computer Presentations**</u>: Obtain a copy of their presentation
    - <u>**Overhead**</u>: Obtain a copy or use the originals and scan them
    - <u>**Blackboard**</u>: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
    - <u>**Handouts**</u>: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack <u>with this completed sheet on top</u> and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to <u>notes@msri.org</u> with the workshop name and your name in the subject line.

Wednesday 2nd talk

scl in mcg

speaker: Mladen Bestvina.

G group. $g \in [G,G]$

$cl(g)$ = smallest k

s.t $g = [\alpha_1, \beta_1], \dots, [\alpha_k, \beta_k]$.

$$scl(g) = \lim_{n\to\infty} \frac{cl(g^n)}{n}$$

Ehdo-Kotschick; Korkmuz

$g \in MCG(\Sigma)$. Dehn twist $\Rightarrow scl(g) > 0$

joint work with Ken Brumberg Koyi Fujiwara

Example: $G = SL_3(\mathbb{Z})$ $cl(g) \leq 48$ $\forall g$. $scl(g) = 0$.

Background.

How to prove $scl(g) > 0$?

Def: A quasi-morphism $f: G \to \mathbb{R}$ is a function s.t

$$\Delta(f) = \sup_{x,y\in G} |f(xy) - f(x) - f(y)| < \infty$$

defect.

Exercise: Every quasi-morphism $f: \mathbb{Z} \to \mathbb{R}$ is bounded dist away from a homomorphism.

Def: $f: G \to \mathbb{R}$ is homogeneous 如 $f(g^n) = n \cdot f(g)$ $\forall g, \forall n$.

$$\hat{f}(g) = \lim_{n\to\infty} \frac{f(g^n)}{n}$$

Criterion: If $\exists f: G \to \mathbb{R}$ quasi-morphism s.t $\hat{f}(g) \neq 0$

then $scl(g) > 0$

$g = \prod_{i=1}^{k} [\alpha_i, \beta_i]$

$\hat{f}(g) \underset{4k\cdot\Delta}{\sim} \sum\left[\hat{f}(\alpha_i) + \hat{f}(\beta_i) + \hat{f}(\alpha_i^{-1}) + \hat{f}(\beta_i^{-1})\right]$

$|\hat{f}(g)| \leq 4k\cdot\Delta$

so $scl(g) > 0$

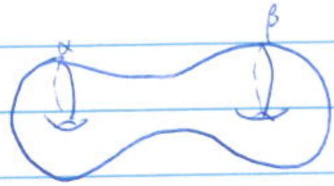
How to prove $scl(g) = 0$?

- If g has finite order, $scl(g) = 0$
- if $g = \gamma g^{-1} \gamma^{-1}$ ("achirel") then $scl(g) = 0$

because $g^2 = g \cdot \gamma g^{-1} \gamma^{-1} = [g, \gamma]$

$g^{2k} = g^k \gamma g^{-k} \gamma^{-1} = [g^k, \gamma]$

- (Endo-Kotschick) $g = g_1 g_2$, $g_2 = \gamma g_1^{-1} \gamma^{-1}$, $g_1 g_2 = g_2 g_1$.



Dehn twist.

$g_1 = D_\alpha$

$g_2 = D_\beta^{-1}$

$g = g_1 \gamma g_1^{-1} \gamma^{-1}$

$g^k = g_1^k \gamma g_1^{-k} \gamma^{-1}$

More generally, $g = g_1 g_2 \cdots g_p$ commuting product.

$g_i^{n_i}$ is conjugate to $g_j^{n_j}$

$\frac{1}{n_1} + \frac{1}{n_2} + \dots + \frac{1}{n_p} = 0 \Rightarrow scl(g) = 0$

- If $g = g_1 \cdots g_p$ commuting product, $scl(g_i) = 0 \Rightarrow scl(g) = 0$

If these cases do not apply, $scl(g) > 0$.

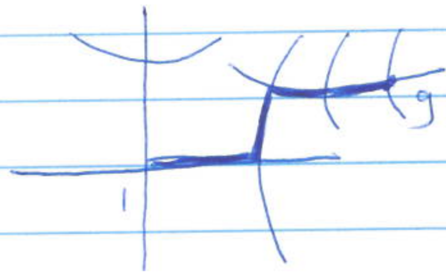
Brooks construction: If $g \in F_n$, $g \neq 1 \Rightarrow scl(g) > 0$

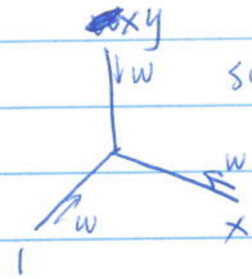
w cyclically reduced word.

$f(g) = (\#$ oriented subwords of g isomorphic to $w) -$

(next page)

(# oriented subwords of g isomorphic to w^{-1})





$\sup |f(xy) - f(x) - f(y)| < \infty$

$w = aba^{-1}b^{-1}$

$f(a^n) = 0 = f(b^n)$

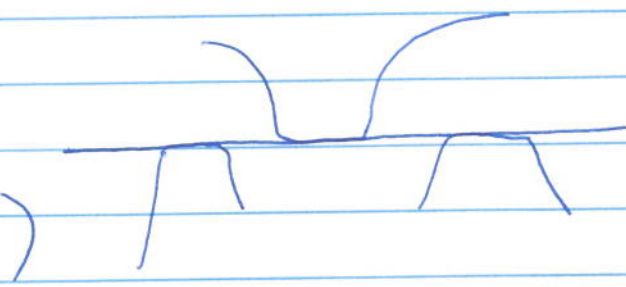
$f(w^n) = n$

Generalization:

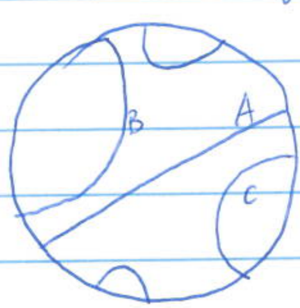
1) $G \curvearrowright T$ tree "discrete axes with no flips"

$g \in G$, $scl(g) > 0$?

If $\gamma(Ax(g)) \neq Ax(g)$, then they have uniform bounded overlaps (projection)



2) Work with quasi-trees instead of trees.



Setting: $\mathcal{Y}$ = collection of metric spaces

$A, B \in \mathcal{Y}$ gives $\pi_B(A) \subset B$ uniform bounded subset.

$d_A(B, C) = \text{diam}(\pi_A(B) \cup \pi_A(C))$

$\exists\, \xi > 0$,

1) At most one of $d_A(B,C)$, $d_B(A,C)$, $d_C(A,B)$ is $> \xi$.

$(*)\begin{cases} 1) \ \ldots\ldots \\ 2) \ \forall A,B,\ \{C \mid d_C(A,B) > \xi\} \text{ is finite.} \end{cases}$

Thm: (BBF) If all $A \in \mathcal{Y}$ are δ-hyperbolic, and $(*)$ holds, then $\exists$ hyperbolic space Y and pairwise disjoint isometric embeddings $A \hookrightarrow Y$ to convex subsets s.t

Nearest point proj of A to B is within bounded distance of $\Pi_B(A)$

- construction is equivalent.
- If every A B a quasi line, then Y is a quasi-tree

Application:

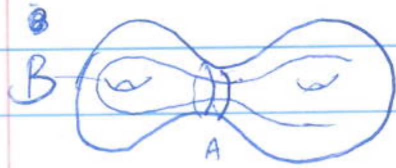
① G hyperbolic, g chirel of infinite order $\Rightarrow scl(g) > 0$ (Epstein - Fujiwara)

$Ax(g) \rightsquigarrow G \curvearrowright \overset{\text{quasi-tree}}{Q} \xrightarrow{\text{Brooks}} \square$

② (Calegari - Fujiwara) g $\overset{\text{pseudo Anosov}}{pA}$, chirel $\Rightarrow scl(g) > 0$.

$M(G\Sigma)$ "$G \curvearrowright$ curve graph of Σ $\rightsquigarrow$ quasi tree

③ Dehn twist,



$\mathcal{C}(A)$ = curve complex of A

$\Pi_A(B)$ Masur-Minsky ~~subspace~~ subsurface projection pA – pseudo Anosov

~~Thm~~

$g \in MCG(\Sigma)$ N–T decompose (of a power)

$g = g_1 g_2 \cdots g_k$, g_i pA on a subsurface of Dehn twist.

g_i is chirel if $g_i^p \not\sim g_i^{-p}$ any $p \neq 0$.

g_i, g_j are chirel, g_i equiv to g_j if $g_i^{n_i} \sim g_j^{n_j}$ some $n_i, n_j \neq 0$

A chirel class is essential if $\Sigma \frac{1}{n_i} \neq 0$

Thm:

$scl(g) > 0 \iff g$ has an essential chirel class.