MSRI

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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Pengcheng Xu **Email/Phone:** pengcheng.xu@okstate.edu

Speaker's Name: David Futer

Talk Title: Surface quotients of hyperbolic buildings

Date: 03/21/2013 **Time:** 2 : 00 am / pm (circle one)

List 6-12 key words for the talk: hyperbolic building; hyperbolic group; surface subgroup; complex of groups

Please summarize the lecture in 5 or fewer sentances: The speaker first introduced the definition of Bourdon's building and gave some examples then answer the question of when there is a discrete subgroup of alltomorphism group such that the quotient is a closed surface of genus g. One consequence of their construction proves a special case of Gromov's surface subgroup conjecture.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Thursday
3rd talk

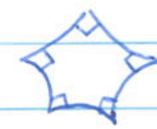
Surface quotients of hyperbolic buildings

speaker : David Futer

joint with Anne Thomas

Goal : Find surface subgroups of hyperbolic groups that act on buildings.

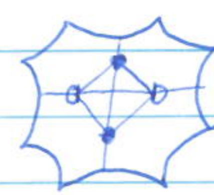
Def: For $p \geq 5$, $v \geq 2$, Bourdon's Building $I(p, v)$, is the unique simply connected 2-complex s.t.
- all 2-cells are right-angled hyperbolic p-gons



- the link of each vertex is the complete bipartite graph $K(v, v)$

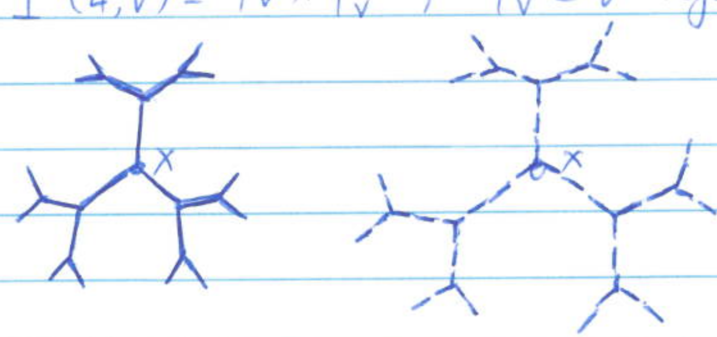
Examples:

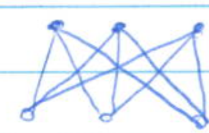
1) $v = 2$, $\Rightarrow I(p, v) = \mathbb{H}^2$, tiled by p-gons



2) (Pseudo-ex) $p = 4$, right-angled squares are Euclidean

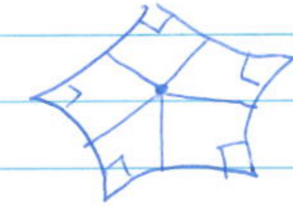
$I(4, v) \cong T_v \times T_v$, $T_v = v$-regular tree.



link(x) = 

Morally: $I(p, v)$ locally looks like $T_v \times T_v$ but is globally a CAT(-1) metric space.

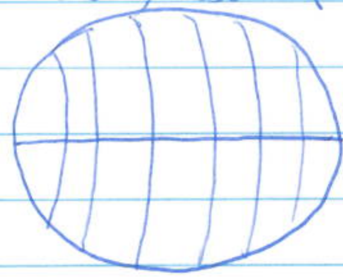
Observations



① . $I(p,v)$ is a CAT(0) square complex

②. When $v \geq 3$, $\text{Isom}(I(p,v))$ is uncountable.

[flexibility]



③ Thm (Bourdon-Dajot, 2000) $I(p,v)$ is QI ...

[Every quasi-isometry is bounded distance from an isometry.

Def: A discrete subgroup $\Gamma \subset \text{Isom}\, I(p,v)$ is called a uniform lattice if $I(p,v)/\Gamma$ is compact.

We'll always assume Γ acts without inversion; i.e. if Γ fixes ~~all~~ cell σ setwise, then it fixes σ pointwise

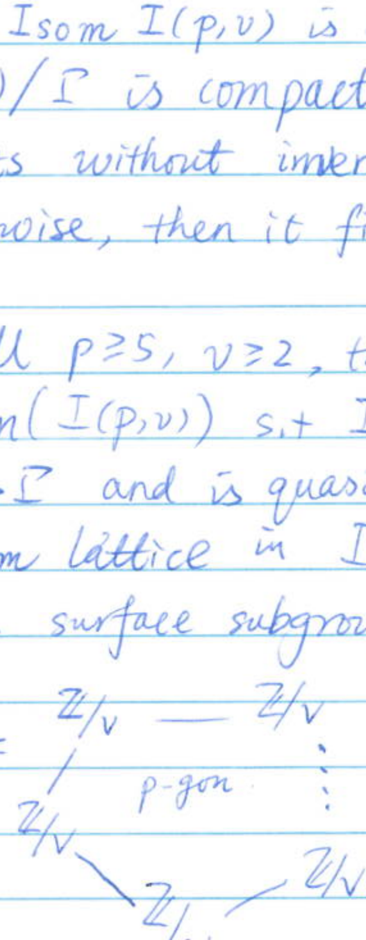
Thm (F-Thomas, 2010) For all $p \geq 5$, $v \geq 2$, there is a uniform lattice $\Gamma \in \text{Isom}(I(p,v))$ s.t. $I(p,v)/\Gamma = \Sigma_{p-3}$

Furthermore, $\pi_1(\Sigma_{p-3}) \hookrightarrow \Gamma$ and is quasi convex.

Corollary (2012) Every uniform lattice in $\text{Isom}(I(p,v))$ contains a quasi convex surface subgroup.

pf of corollary:

Step 1: Consider $\Gamma_0(p,v) =$



$\Gamma_0(p,v)$ acts on $I(p,v)$, with quotient a single p-gon.

Haglund (2006): For any other $\Gamma \in Aut(I(p,v))$, uniform lattice, we have Γ_0 commensurable to Γ if Γ is QCERF.
[quasi convex subgroups of Γ are separable]

Step 2: Cubulate! Γ is a hyperbolic, cubulated group $\underset{\text{[Agol-Wise]}}{\Longrightarrow}$ Γ is virtually special

$\underset{\text{Haglund-Wise}}{\Longrightarrow}$ Γ is QCERF.

Step 3: apply our theorem to get one lattice with QC surface subgroup. $\square$

Remark: Kim (2010) $\Gamma_0(p,v)$ contains QC surface subgroup if $p \geq 5$.

Holt-Rees (2012) $\Gamma_0(p,v)$ contains surface subgroup if $p \geq 3$

Q: For what triples (p,v,g) does there exist a uniform lattice $\Gamma \subset \mathrm{Isom}\, I(p,v)$ s.t. $I(p,v)/\Gamma = \Sigma_g$, surface of genus g?

One thing that is needed is for S_g to be tiled by right-angled p-gons.

Lemma: Σ_g can be tiled by F copies of a right-angled p-gon $\Longleftrightarrow F(p-4) = 8(g-1)$

pf: ($\Rightarrow$) hyperbolic area

$$\text{Area}(p\text{-gon}) = \frac{\pi}{2}(p-4) \qquad \text{Area}(\Sigma_g) = 8\pi(g-1)$$

(or use Euler Characteristic)

($\Leftarrow$) direct construction

Refined question: for what triples (p, v, F) does there exist $\Gamma = \Gamma_{p,v,F}$ s.t. $I(p,v)/\Gamma$ is a surface tiled by F, right-angled p-gon?
When this happens, we say $\Gamma_{p,v,F}$ exists.

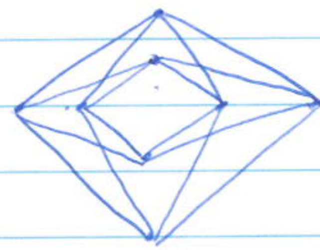
Thm (F-Thomas 2010)
1. If v is even, then $\forall p, F.\ \exists \Gamma_{p,v,F}$
2. If F is divisible by 4, then $\forall v, p,\ \exists \Gamma_{p,v,F}$
   (for previous theorem, take $F=8 \Rightarrow F(p-4) = 8\underbrace{(p-4)}_{g-1}$)
3. If F is composite and v is divisible by 15, then $\forall p$, $\exists \Gamma_{p,v,F}$.
4. If F is odd, $v=q^n$, q odd prime, then $\forall p\ \nexists \Gamma_{p,v,F}$

Pf idea for existence results.
Construct Γ as a complex of finite groups.
Start with surface a Σ_g, tiled by p-gons. Assign a finite group G_σ to every cell σ. [Meaning: in the universal cover, $I(p,v)$, G_σ is the stabilizer of a lift $\tilde{\sigma}$ under the group Γ.]
Whenever $\sigma \subset \tau$, we want $G_\tau \subset G_\sigma$.

How to do this when v is even? (1).
Let $r = v/2$. Label every face of the tiling by $\{1\}$, every edge by $\mathbb{Z}/r$, every vertex $(\mathbb{Z}/r)$?

$$\begin{tikzcd}
\{1\} \arrow[r] \arrow[d] & \mathbb{Z}/r \arrow[d] & \{1\} \arrow[l] \arrow[d] \\
\mathbb{Z}/r \arrow[r] & \mathbb{Z}/r \times \mathbb{Z}/r & \mathbb{Z}/r \arrow[l] \\
\{1\} \arrow[r] \arrow[u] & \mathbb{Z}/r \arrow[u] & \{1\} \arrow[l] \arrow[u]
\end{tikzcd}$$

picture of the link ~~of~~ of a vertex (for $r=2$)



Now, develop this local picture to its universal cover. The development is possible because we have a negatively curved complex of groups.
[Casson, Gersten-Stallings, Haefliger, ~1990] □

For Thm ②, it's the same idea, but not every tiling will work.

Prop: Let Y be a tessellation of Σ_g by right-angled p-gons. Then TFAE:

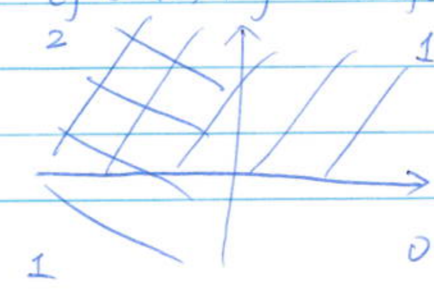
① $\forall\, v \geq 2$, $\exists$ complex of finite groups over Y whose universal cover is $I(p,v)$

② The ~~edges~~ geodesics $h_1, \ldots, h_n$ of the tiling can be oriented s.t. $\sum [h_i] = 0 \in H_1(\Sigma_g, \mathbb{Z})$

Furthermore, ① and ② are impossible when F is not divisible by 4.

Sketch proof:

($\Leftarrow$) Assume $\sum [h_i] = 0 \in H_1(\Sigma_g)$ Then $\sum [h_i] = \partial \sum c_j F_j$
$c_j \in \mathbb{Z}$, F_j are faces. May assume $c_j \geq 0$



$\boxed{b = v-1}$

$$\begin{tikzcd}
\mathbb{Z}/b \times \mathbb{Z}/b \arrow[r] \arrow[d] & \mathbb{Z}/b & \mathbb{Z}/b \arrow[d] & \mathbb{Z}/b \arrow[l] \arrow[d] \\
\mathbb{Z}/b \times \mathbb{Z}/b \arrow[r] & \mathbb{Z}/b & \mathbb{Z}/b & \mathbb{Z}/b \arrow[l] \\
\mathbb{Z}/b \arrow[u] \arrow[r] & \mathbb{Z}/b \arrow[u] & \{1\} \arrow[l] \arrow[ur] &
\end{tikzcd}$$

($\Rightarrow$) Reverse-engineer this construction. If face F_i is assigned finite group G_i then consider $\Sigma \log |G_i| \cdot F_i \in C^2(\Sigma_g, \mathbb{R})$ and taken the boundary of this 2-chain. Need to consider intersection pairings. $\square$