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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Pengcheng Xu Email/Phone: pengcheng.xu@okstate.edu

Speaker's Name: Anne Thomas

Talk Title: Polyhedral complexes, lattices and surface subgroups

Date: 03 / 21 / 2013 Time: 3 : 30 am / pm (circle one)

List 6-12 key words for the talk: Locally compact groups; lattices; polyhedral complexes; CAT(0) cube complexes; hyperbolic groups

Please summarize the lecture in 5 or fewer sentances: The speaker started with examples of polyhedral complexes which may be natually equipped with a metric so that they become cube complexes then survey the known construction of cocompact lattices. Then she focused on the case of X a right angled building and discuss the construction of cocompact lattices in complete Kac-Moody groups which have building X.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
    - **Computer Presentations**: Obtain a copy of their presentation
    - **Overhead**: Obtain a copy or use the originals and scan them
    - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
    - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Lattices, polyhedral complexes, cubulations and surface subgroups

Anne Thomas

MSRI Hot Topics – Surface subgroups and cube complexes
21 March 2013

Introduction

2007: AIM workshop "Problems in geometric group theory"

2007–2008: Benson Farb, Chris Hruska and I wrote "Problems on automorphism groups of nonpositively curved polyhedral complexes and their lattices". Refereed by Haglund.

Today:

1. Locally compact groups G and their lattices
2. Polyhedral complexes X
3. Applications of Agol's Theorem to lattices in $G = \mathrm{Aut}(X)$
4. Recent result with Inna Capdeboscq on Kac–Moody lattices with surface subgroups

Locally compact groups

G locally compact topological group with Haar measure μ

Examples

1. $G = (\mathbb{R}^n, +)$ with Lebesgue measure
2. $G = \mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$

Lattices

G locally compact, Haar measure μ

A subgroup $\Gamma < G$ is a lattice if

- Γ is discrete
- $\mu(\Gamma \backslash G) < \infty$ (finite covolume)

A lattice $\Gamma < G$ is

- uniform or cocompact if $\Gamma \backslash G$ is compact
- otherwise, non-uniform or non-cocompact

Example

Let $G = \mathrm{PSL}_2(\mathbb{C})$ and let $\Gamma < G$ be a torsion-free lattice.

- If Γ is uniform, Γ is the fundamental group of a closed hyperbolic 3-manifold.
- If Γ is non-uniform, Γ is the fundamental group of a finite volume non-compact hyperbolic 3-manifold.

If Γ has torsion replace manifold by orbifold.

Polyhedral complexes

A polyhedral complex is a CW complex obtained by gluing together convex polyhedra by isometries along their edges.

All polyhedra from fixed constant curvature space: $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$.

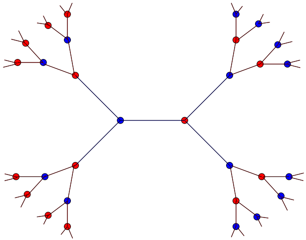
A polygonal complex is a 2-dimensional polyhedral complex.

Examples

1. Trees
2. Products of trees
3. Davis complexes
4. Buildings

Trees

T locally finite tree



$G = \mathrm{Aut}(T)$ is a locally compact group

G non-discrete $\iff \exists \{g_n\} \subset G$, $g_n \neq 1$, so that g_n fixes $\mathrm{Ball}(n)$

Example

$G = \mathrm{Aut}(T)$ nondiscrete for $T = T_m$ the m-regular tree, $m \geq 3$

Lattices in Aut(T)

T locally finite tree
$G = \mathrm{Aut}(T)$ with Haar measure μ

$\Gamma < G$ is discrete $\iff$ Γ acts with finite stabilisers

Theorem (Serre)

Can normalise μ so that $\forall$ discrete $\Gamma < G$,

$$\mu(\Gamma \backslash G) = \sum_{v \in T/\Gamma} \frac{1}{|Stab_\Gamma(\overline{v})|}$$

and Γ uniform $\iff$ T/Γ compact.

Uniform tree lattices

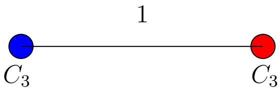
$\Gamma < \mathrm{Aut}(T)$ is a uniform lattice
$\iff$ Γ acts on T with finite quotient and finite stabilisers
$\iff$ Γ is fundamental group of a finite graph of finite groups with universal cover T.

Theorem (Bass–Kulkarni 1991)

Uniform tree lattices are virtually free.

Example

$G = \mathrm{Aut}(T_3)$
$\Gamma = \pi_1(\text{graph of groups}) \cong C_3 * C_3$ is uniform lattice in G
$\mu(\Gamma \backslash G) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

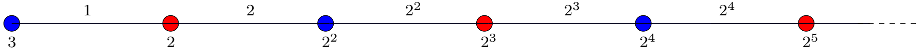


Non-uniform tree lattices

$\Gamma < \text{Aut}(T)$ is a non-uniform lattice
$\iff$ Γ acts on T with infinite quotient and finite stabilisers growing "fast enough"
$\iff$ Γ is fundamental group of an infinite graph of finite groups with universal cover T and the vertex groups growing "fast enough"

Example

$G = \text{Aut}(T_3)$
$\Gamma = \pi_1$(graph of groups) is non-uniform lattice in G
$\mu(\Gamma \backslash G) = \frac{1}{3} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{4}{3}$



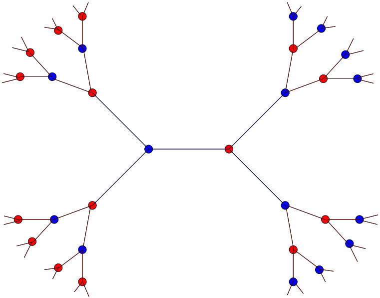
Tree lattices

Program (Bass, Lubotzky, . . .)

Compare lattices in Aut(T) *to lattices in Lie groups.*

Motivation:

- Study lattices in Lie groups via action on symmetric space
  e.g. upper half-plane is symmetric space for $\mathrm{SL}_2(\mathbb{R})$
- Study lattices in algebraic groups over nonarchimedean local fields via action on building
  e.g. T_{q+1} is building for $\mathrm{SL}_2(\mathbb{F}_q((t)))$



Lattices in $\mathrm{Aut}(X)$

X locally finite polyhedral complex
$G = \mathrm{Aut}(X)$ is locally compact, has Haar measure μ

$\Gamma < G$ is discrete $\iff$ Γ acts with finite stabilisers

Theorem (Serre)

Can normalise μ so that $\forall$ discrete $\Gamma < G$,

$$\mu(\Gamma\backslash G) = \sum_{v \in X/\Gamma} \frac{1}{|Stab_\Gamma(\overline{v})|}$$

and Γ uniform $\iff$ X/Γ compact.

Cubulating uniform lattices in Aut(X)

X locally finite polyhedral complex
$G = \mathrm{Aut}(X)$

$\Gamma < G$ is a uniform lattice $\iff$ Γ acts properly discontinuously and cocompactly on X

Let $\Gamma < G$ be a uniform lattice.

1. By definition, if X is CAT(0) then Γ is a CAT(0) group.
2. By Milnor–Svarc Lemma, if X is CAT(-1) or δ-hyperbolic then Γ is hyperbolic.
3. By Agol's Theorem, if X is a δ-hyperbolic cube complex then Γ is virtually special.

The link condition for polygonal complexes

X polygonal complex

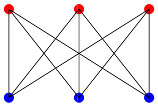
Metrise link $\mathrm{Lk}(v, X)$ so edge length = angle at v.

Gromov Link Condition:

1. If X is piecewise Euclidean and all embedded circuits in all links have length $\geq 2\pi$, then X is locally CAT(0).
2. If X is piecewise hyperbolic and all embedded circuits in all links have length $\geq 2\pi$, then X is locally CAT(-1).

Example

Product of trees is CAT(0)



Caution: Property (T)

Theorem (Cartwright, Młotkowski and Steger 1994, Żuk 1996, Ballmann and Świątkowski 1997, Dymara and Januszkiewicz 2002)

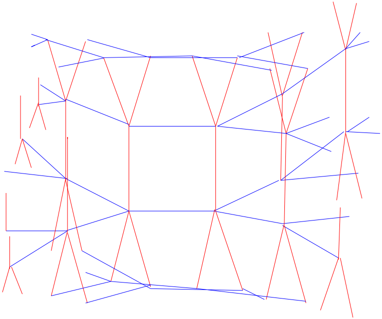
For certain simplicial X, $G = \mathrm{Aut}(X)$ has Property (T).

G has (T)

$\implies$ every lattice in G has (T)

$\implies$ no lattice in G can be cubulated.

Caution: products of trees



Theorem (Burger–Mozes, 2002)

Suppose $X = T_p \times T_p$ *with* p *prime, and* $G = \mathrm{Aut}(X)$. *There is a torsion-free uniform lattice* $\Gamma < G$ *which is a simple group.*

Corollary

There is a NPC finite square complex with no finite-sheeted covers.

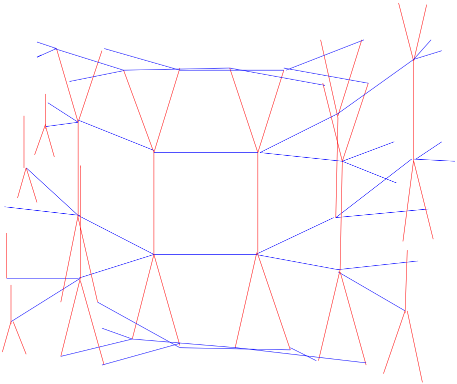
(k, L)-complexes

Given $k \geq 3$ and a graph L, a (k, L)-complex is a polygonal complex such that

1. every face is a regular k-gon
2. the link of every vertex is L

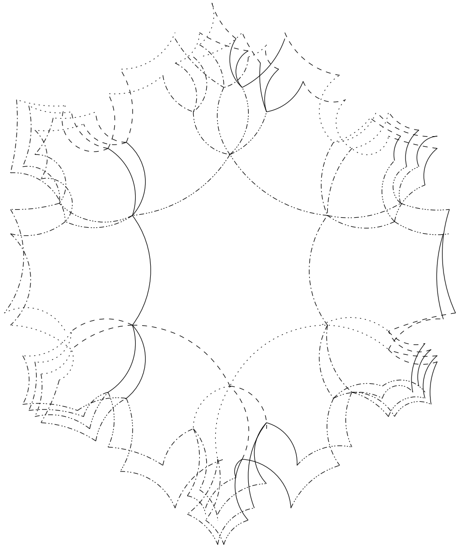
Examples of (k, L)-complexes

- Product of trees: $k = 4$, $L = K_{m,n}$



Examples of (k, L)-complexes

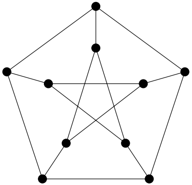
- Bourdon's building: $k \geq 5$, $L = K_{v,v}$



Examples of (k, L)-complexes

Theorem (Świątkowski 1998)

Let $k \geq 4$ *and* L *be the Petersen graph. Then there is a unique CAT(0)* (k, L)*-complex* X*. Moreover,* $\mathrm{Aut}(X)$ *is nondiscrete and* $\mathrm{Aut}(X)$ *acts transitively on flags in* X*.*



A flag in a polygonal complex X is a triple (v, e, f) where vertex v is contained in edge e is contained in face f.

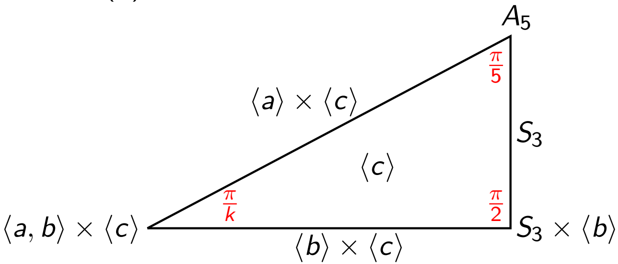
Lattices on (k, L)-complexes

$k \geq 4$, L the Petersen graph
X the unique CAT(0) (k, L)-complex

Work in progress with Inna Capdeboscq and Michael Giudici: constructing flag-transitive uniform lattices in $\mathrm{Aut}(X)$ as fundamental groups of triangles of groups (Gersten–Stallings).

Example

$\langle a \rangle \cong \langle b \rangle \cong \langle c \rangle \cong C_2$
$\langle a, b \rangle \cong D_{2k}$
$A_5 \leq S_5 = \mathrm{Aut}(L)$

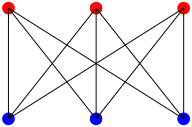


Turning (k, L)-complexes into square complexes

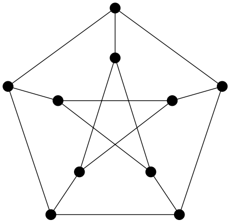
The girth of a graph L is the number of edges in a shortest embedded circuit.

Examples

1. $K_{m,n}$ has girth 4

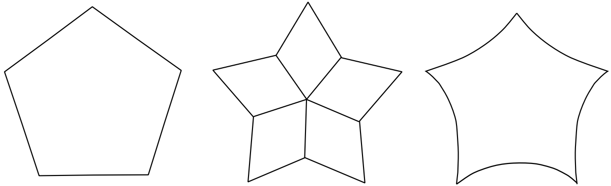


2. Petersen graph has girth 5



Turning (k, L)-complexes into square complexes

A k-gon can be metrised as a Euclidean k-gon, a cycle of k Euclidean squares, or a hyperbolic k-gon:



Theorem (Gromov)

Let X be a simply connected (k, L)-complex. Let $g = \mathrm{girth}(L)$. If $k \geq 4$ and $g \geq 5$, or $k \geq 5$ and $g \geq 4$, then X can be metrised as a square complex which is δ-hyperbolic.

Turning (k, L)-complexes into square complexes

Let X be a simply connected (k, L)-complex and $g = \text{girth}(L)$.

Corollary

If $k \geq 4$ and $g \geq 5$, or $k \geq 5$ and $g \geq 4$, a uniform lattice $\Gamma < \text{Aut}(X)$ is virtually special.

So Γ is linear, residually finite, has separable quasi-convex subgroups, is virtually torsion free, large, ...

Remarks

1. Earlier work of Wise:
   - $X = X(k, L)$ as a square complex is a $\mathcal{VH}$-complex $\iff$ k is even and L is bipartite
   - Γ the fundamental group of a negatively curved k-gon of finite groups, $k \geq 4$
2. Uniform lattices in $\text{Aut}(X)$ often have torsion, unlike e.g. fundamental groups of 3-manifolds.

Davis complexes

Fix $m \geq 2$ and L a simplicial graph

Define $W = W(m, L)$ to be the Coxeter group with

- generating set $S = \text{Vert}(L)$
- relations
  - $s^2 = 1$ for all $s \in S$
  - $(st)^m = 1 \iff s$ and t are adjacent in L

Remarks

1. W has presentation

$$W = \langle S \mid s^2 = 1 \, \forall \, s \in S, (st)^{m_{st}} = 1 \rangle$$

   where $m_{st} \in \{m, \infty\}$, $m_{st} = m \iff s$ and t are adjacent in L
2. if $m = 2$ then W is a RACG
3. if s and t are adjacent, $\langle s, t \rangle \cong D_{2m}$

Davis complexes

If $m = 2$ assume girth$(L) \geq 4$.

The Davis complex $X = X(m, L)$ for $W = W(m, L)$ is the 2-complex with:

- 1-skeleton the Cayley graph of W w.r.t. S
- a $2m$-gon glued along each circuit with edge labels $\underbrace{s, t, s, t, \ldots}_{2m}$

X is a (k, L)-complex with $k = 2m \geq 4$.
W acts on X cocompactly with finite stabilisers.

Theorem (Gromov, Davis, Moussong)

1. *If the faces of X are metrised as regular Euclidean $2m$-gons then X is CAT(0).*
2. *X may be metrised as a δ-hyperbolic square complex provided if $m = 2$ then girth$(L) \geq 5$ and if $m \geq 3$ then girth$(L) \geq 4$.*

Lattices on Davis complexes

$X = X(m, L)$ the Davis complex for $W = W(m, L)$

Theorem (Haglund–Paulin 1998, White 2012)

$\mathrm{Aut}(X)$ *is nondiscrete* $\iff$ L *is "flexible"*

W is a uniform lattice in $G = \mathrm{Aut}(X)$.

$\Gamma_1, \Gamma_2 < G$ are commensurable (up to conjugacy in G) if for some $g \in G$, $\Gamma_1 \cap \Gamma_2^g$ have a common finite index subgroup.

Theorem (Haglund 2006)

Suppose X is δ-hyperbolic. If a uniform lattice $\Gamma < \mathrm{Aut}(X)$ has separable quasiconvex subgroups, then Γ is commensurable to W.

Again using Agol's Theorem:

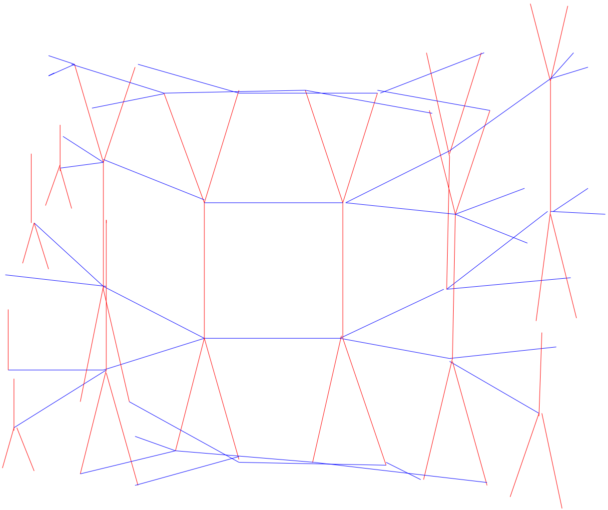
Corollary

All uniform lattices in $\mathrm{Aut}(X)$ are commensurable.

Buildings

Examples

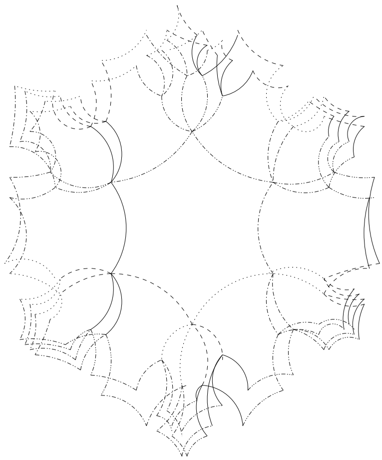
- Product of trees: apartments are tessellated Euclidean planes



Buildings

Examples

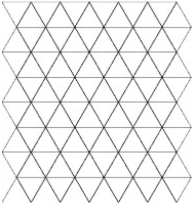
- Bourdon's building: apartments are tessellated hyperbolic planes



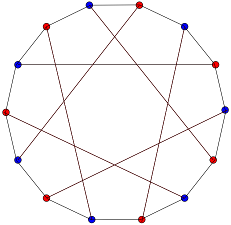
Buildings

Examples

- Building for $SL_3(\mathbb{F}_2((t)))$ has apartments



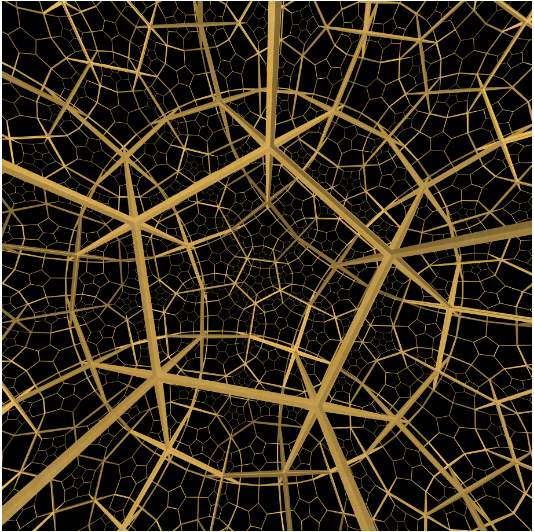
and links



Buildings

Examples

- There are 3-dimensional hyperbolic buildings with apartments



Right-angled buildings

Data:

1. L a simplicial graph, $S = \text{Vert}(L)$
2. $(q_s)_{s \in S}$, with $q_s \geq 2$

Let Γ_0 be the graph product of cyclic groups of order q_s over L.

There is a locally finite cube complex X, called a right-angled building, such that Γ_0 is the "standard uniform lattice" in $\text{Aut}(X)$.

Theorem (Gromov, Moussong, Davis)

1. *X is a CAT(0) cube complex*
2. *X is δ-hyperbolic $\iff$ L has no empty squares.*

Lattices on right-angled buildings

X right-angled building with data L, (q_s)

Corollary

If L has no empty squares, every uniform lattice $\Gamma < \mathrm{Aut}(X)$ is virtually special.

Example

If G_s any finite group of order q_s and Γ is the graph product of the G_s over L, then Γ is a uniform lattice in $\mathrm{Aut}(X)$. Residual finiteness and linearity of Γ: Hsu–Wise.

Lattices on right-angled buildings

X right-angled building with data L, (q_s)
Γ_0 graph product of $\mathbb{Z}/q_s\mathbb{Z}$ over L

Theorem (Haglund 2006)

Suppose X is δ-hyperbolic. If a uniform lattice $\Gamma < \mathrm{Aut}(X)$ has separable quasiconvex subgroups, then Γ is commensurable to Γ_0.

Corollary

If L has no empty squares, all uniform lattices in $\mathrm{Aut}(X)$ are commensurable.

Januszkiewicz–Świątkowski proved graph product $\Gamma = \Gamma(G_s)$ commensurable to Γ_0.

Lattices in Kac–Moody groups

Let G be a complete Kac–Moody group over $\mathbb{F}_q$
e.g. $G = \mathrm{SL}_n(\mathbb{F}_q((t)))$

G has a building X but G is much smaller than $\mathrm{Aut}(X)$

Theorem (Rémy 1999, Carbone–Garland 2003)

For q large enough, G admits a nonuniform lattice.

Both proofs start with a subgroup of G and show it is a nonuniform lattice by considering the action on X.

Recent result

Let G be a complete Kac–Moody group over $\mathbb{F}_q$

Theorem (Capdeboscq–T 2012)

Assume that the building X for G is right-angled. Then G admits a uniform lattice Γ in the following cases:

1. *q even and $q \equiv 3 \pmod 4$*
2. $q \equiv 1 \pmod 4$ *and the building for G is $I_{p,q+1}$*

Moreover Γ contains a surface subgroup.

We start with a uniform lattice $\Gamma < \mathrm{Aut}(X)$

1. $\Gamma = \Gamma_0$ graph product of finite cyclic groups
2. Γ a lattice in $\mathrm{Aut}(I_{p,q+1})$ with surface quotient [Futer–T 2012]

Then use covering theory for complexes of groups, i.e. check local injectivity, to embed Γ in G.
In both cases Γ has a surface subgroup [Kim 2012, Holt–Rees 2012, Futer–T 2012].