

Overview - Peter Scholze

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Summary: This talk gives an overview of the development of the theory of perfectoid spaces. Scholze describes three “phases” of study, applying them to different topics in number theory. The first phase was giving a correspondence between geometry in characteristic zero and characteristic p , with the goal of proving Deligne’s weight-monodromy conjecture. The second phase was studying p -adic Hodge theory, and how it varies in families. The third phase discussed here is the realization of important special cases of “infinite-type rigid geometry” via perfectoid spaces.

Work on perfectoid spaces got started from a suggestion of Rapoport, that it should be possible to reduce the weight-monodromy conjecture to equal characteristic after a “highly ramified base-change”.

Conjecture 1 (Weight-Monodromy Conjecture (Deligne, 1970)). *If $X/\mathbb{Q}_p$ is a proper smooth variety, $i \geq 0$, and $\ell \neq p$, then we have étale cohomology groups $H^i(X_{\overline{\mathbb{Q}_p}}, \overline{\mathbb{Q}_\ell})$ with their natural Galois action. These groups have two natural filtrations, the weight filtration and monodromy filtration. Deligne’s conjecture is that these are equal.*

Deligne showed that the corresponding conjecture for $X/\mathbb{F}_p((t))$ is true in his “Weil II” paper. The case over $\mathbb{Q}_p$ is possibly the most important open problem about étale cohomology.

One way to interpret “highly-ramified base change” is as in the following theorem.

Theorem 2 (Fontaine-Wintenberger, 1970’s). *Let $K = \mathbb{Q}_p(p^{1/p^\infty})$, i.e. $\mathbb{Q}_p$ with all p -power roots of p adjoined (an infinitely ramified extension of $\mathbb{Q}_p$). Then*

$$\mathrm{Gal}(\overline{\mathbb{F}_p((t))}/\mathbb{F}_p((t))) \cong \mathrm{Gal}(\overline{\mathbb{Q}_p}/K) \subseteq \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p).$$

So after this base-change, Galois theory over $\mathbb{Q}_p$ is the same as over $\mathbb{F}_p((t))$! One can then ask the natural question of whether, given a variety X/K , one can realize the G_K -representation $H^i(X_{\overline{K}}, \overline{\mathbb{Q}_\ell})$ as $H^i(X'_{\overline{\mathbb{F}_p((t))}}, \overline{\mathbb{Q}_\ell})$ for some

$X'/\mathbb{F}_p((t))$, in a way such that the Galois actions are compatible via the isomorphism of the theorem. If so, then Deligne's proof of the weight-monodromy conjecture over $\mathbb{F}_p((t))$ implies the conjecture over $\mathbb{Q}_p$. In other words, does the relation between K and $\mathbb{F}_p((t))$ extend to a correspondence of higher-dimensional objects $X \leftrightarrow X'$? Studying this led to “Phase 1” of the study of perfectoid spaces.

Phase 1 The first case of such a correspondence we'd want to understand is how $\mathbb{P}_K^n$ corresponds to $\mathbb{P}_{\mathbb{F}_p((t))}^n$. There's one important difference between these two objects: the behavior of the Frobenius map $\varphi : \mathbb{P}^n \rightarrow \mathbb{P}^n$ given by $[x_0 : \dots : x_n] \mapsto [x_0^p : \dots : x_n^p]$. In characteristic p , φ is a homeomorphism of underlying topological spaces on the étale site, but not in characteristic zero!

To get the Fontaine-Wintenberger theorem, we had to adjoin all of the p -power roots of p to $\mathbb{Q}_p$. We'll have to do something similar here. So we can refine the question of the above proposition and ask whether there is a correspondence

$$\varprojlim_{\varphi} \mathbb{P}_K^n \longleftrightarrow \varprojlim_{\varphi} \mathbb{P}_{\mathbb{F}_p((t))}^n \cong \mathbb{P}_{\mathbb{F}_p((t))}^n.$$

The answer to this is yes! In a suitable setup, we have bijections between the underlying sets of points and of étale sites,

$$\left| \varprojlim_{\varphi} \mathbb{P}_K^n \right| \cong \left| \mathbb{P}_{\mathbb{F}_p((t))}^n \right| \quad \left(\varprojlim_{\varphi} \mathbb{P}_K^n \right)_{\text{ét}} \cong \left(\mathbb{P}_{\mathbb{F}_p((t))}^n \right)_{\text{ét}}.$$

So this gives the correspondence for projective spaces. But the weight-monodromy conjecture for $\mathbb{P}^n$ is trivial, so we need some more examples. One way we can produce these is by considering a dynamical system (X, φ) in characteristic zero with X/K and φ a lift of Frobenius. In this case there will exist $X'/\mathbb{F}_p((t))$ corresponding to $\varprojlim_{\varphi} X$.

Specific example of this: Let X be the canonical lift of an ordinary elliptic curve (or an abelian variety), and φ the canonical lift of Frobenius. This φ map has the nice property that it's finite étale in characteristic zero.

So the diagram to keep in mind is the following. We want to compare varieties over K with varieties over $\mathbb{F}_p((t))$, but we can't do that directly. Instead, we'll compare “perfectoid spaces” over K and “perfectoid spaces” over $\mathbb{F}_p((t))$, which are rigid-analytic objects that are actually equivalent to each other (via the *tilting equivalence*). For a variety X' over $\mathbb{F}_p((t))$, there's a canonical way to pass to a perfectoid space $\varprojlim_{\text{Frob}} X'$. In characteristic zero, there's a way to pass from a pair (X, φ) to a perfectoid space $\varprojlim_{\varphi} X$.

But there's no canonical way to pass from a variety over characteristic zero by itself to a perfectoid space! Moreover, while we found lifts of Frobenius for $\mathbb{P}_K^n$ and for elliptic curves, most varieties X don't admit one (for instance curves of genus ≥ 2). Way out of this: embed everything in projective space! Take $X = X_0 \hookrightarrow \mathbb{P}_K^n$, and then pull back our canonical tower for projective space

$$\dots \longrightarrow \mathbb{P}_K^n \xrightarrow{\varphi} \mathbb{P}_K^n \xrightarrow{\varphi} \mathbb{P}_K^n$$

to a tower

$$\cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0.$$

Then $\varprojlim X_i \subseteq \varprojlim \mathbb{P}_K^n \hookrightarrow \mathbb{P}_{\mathbb{F}_p((t))}^n$ is a perfectoid space! But it does *not* tilt to a variety over $\mathbb{F}_p((t))$. Example: The curve

$$X_0 = \{x_0 + x_1 + x_2 = 0\} \subseteq \mathbb{P}_K^2$$

pulls back to

$$X_n = \{x_0^{p^n} + x_1^{p^n} + x_2^{p^n} = 0\},$$

which are Fermat curves of genus going to ∞ . Then $\varprojlim X_n$ is an object that we can think of living in $\mathbb{P}_{\mathbb{F}_p((t))}^2$, but it's a very badly-behaved one: for instance, it meets lines at infinitely many points! We can think of it as a fractal corresponding to the dynamical system $(\mathbb{P}^2, \varphi)$. However, we can get around this issue in some cases via the following lemma:

Lemma 3. *If $X \subseteq \mathbb{P}^n$ is a complete intersection, the fractal we get can be approximated by algebraic varieties.*

By using the lemma, you can relate the cohomology of X to the cohomology of the approximating varieties, and obtain the weight-monodromy conjecture for complete intersections.

Phase 2 The next direction of study of perfectoid spaces was p -adic Hodge theory. This is where studying perfectoid spaces most naturally belongs; most of the ideas came from p -adic Hodge theory of Faltings, Kedlaya, and others. The basic idea (due to Faltings, 1990): Cover varieties locally by perfectoid spaces. Have $X/\mathbb{Q}_p$, and analyze by considering opens $U \hookrightarrow X$ (in the sense of rigid analytic geometry), taking a pro-finite étale cover $\tilde{U} \rightarrow U$ with $\tilde{U}$ perfectoid.

Example: For $\mathbb{P}^n$, take the open set $\mathbb{G}_m^n$, and take the cover $\tilde{\mathbb{G}}_m^n = \varprojlim_{\varphi} \mathbb{G}_m^n$.

This idea of “extracting p -power roots” has been used in p -adic Hodge theory for a long time. Given these covers $\tilde{U} \rightarrow U$ of open sets, could think to tilt the perfectoid spaces $\tilde{U}$ to equal characteristic, but then can't glue them to an algebraic variety. So we forget about tilting in this setup - perfectoid spaces still have useful properties that help us in this setup!

What are some of these properties? First, they have very small $\mathbb{F}_p$ -étale cohomology (occurs only in degrees 0 and 1 for affinoids). Can refine this and say that perfectoid spaces are “almost contractible” via the “almost purity theorem”, a key technical component of Faltings' approach to p -adic Hodge theory. The theory of perfectoid spaces actually lets you prove a more general version of the almost purity theorem than before, and this is then the main tool used when we apply perfectoid spaces to p -adic Hodge theory!

Remark: We can make sense of $\Omega_{R/\mathbb{F}_1}^1$, i.e. “differentials over the field of one element”, in this setup, The idea is that $\mathbb{Z}/\mathbb{F}_1$ is dimension 1 and smooth, so $\Omega_{\mathbb{Z}/\mathbb{F}_1}^1 \cong \mathbb{Z}$. So we should have an exact sequence

$$0 \rightarrow \Omega_{\mathbb{Z}/\mathbb{F}_1}^1 \otimes R \cong R \rightarrow \Omega_{R/\mathbb{F}_1}^1 \rightarrow \Omega_{R/\mathbb{Z}}^1 \rightarrow 0.$$

In particular, let R be a smooth $\mathbb{Q}_p$ -Tate algebra.

Proposition 4 (“Faltings’ extension”). *For any perfectoid R -algebra $\tilde{R}$, there exists an object we can think of as “ $\Omega_{R/\mathbb{F}_1}^1 \otimes_R \tilde{R}$ ” that’s compatible with base change in $\tilde{R}$ as expected and sits in an exact sequence*

$$0 \rightarrow \tilde{R}(1) \rightarrow \Omega_{R/\mathbb{F}_1}^1 \otimes_R \tilde{R} \rightarrow \Omega_{\tilde{R}/\mathbb{Q}_p}^1 \rightarrow 0.$$

Phase 3 This phase started March 22, 2011, at a conference at Princeton. At 14:30 Scholze gave the first big talk on perfectoid spaces, and at 16:00 Jared Weinstein gave equations for the Lubin-Tate tower at infinite level,

$$\mathbb{Z}_p[[X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}]]/\Delta.$$

It was evident that what he was describing was a perfectoid space, and that perfectoid spaces arose naturally in contexts like this!

Brief description of the Lubin-Tate tower. This is a tower

$$\cdots \rightarrow \mathcal{M}_2 \rightarrow \mathcal{M}_1 \rightarrow \mathcal{M}_0$$

where $\mathcal{M}_0$ is an $(n-1)$ -dimensional open unit ball, and $\mathcal{M}_k/\mathcal{M}_0$ is a $\mathrm{GL}_n(\mathbb{Z}/p^k\mathbb{Z})$ -torsor. The whole picture should be a $\mathrm{GL}_n(\mathbb{Z}_p)$ -torsor, so people wanted to make sense of the inverse limit. But if you do that, you lose all finiteness statements, and there wasn’t any corresponding definition of rigid-analytic space. One way to try to get around this is to introduce integral models, but that’s very complicated and non-canonical. But what Weinstein proved was that $\mathcal{M}_\infty = \varprojlim \mathcal{M}_k$ exists as a perfectoid space, and in some sense this infinite-level object is simpler than the finite-level ones!

One reason to consider these things at infinite level: the Drinfeld tower. Start with Drinfeld’s upper half-space $\mathcal{N}_0$, which is $\mathbb{P}^{n-1}$ with all $\mathbb{Q}_p$ -rational hyperplanes removed. There’s a tower

$$\cdots \rightarrow \mathcal{N}_2 \rightarrow \mathcal{N}_1 \rightarrow \mathcal{N}_0,$$

with an action of $\mathcal{O}_D^\times$. Faltings proved that at infinite level, this is isomorphic to the Lubin-Tate tower! But without a good framework it was hard to even make sense of what that meant; Fargues worked it out more carefully. But one can take $\mathcal{N}_\infty = \varprojlim \mathcal{N}_k$ as a perfectoid space, and Scholze-Weinstein proves that this is isomorphic to $\mathcal{M}_\infty$.

So there are naturally examples of perfectoid spaces arising as a framework for (very special) “infinite-type rigid geometry”. Also, again in this situation, tilting doesn’t seem very useful.

Other examples of the same phenomenon: Shimura varieties at infinite level, closely related to completed cohomology of Emerton. Another example is an abelian variety A , with the multiplication-by- p map $A \rightarrow A$; then $\varprojlim A$ is perfectoid.