MSRI Hot Topics Workshop: Perfectoid Spaces and their Applications

Adic Spaces I - Eugen Hellmann

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Summary: This lecture defined adic spaces, which we will use as the geometric framework for perfectoid spaces. Adic spaces are one system for rigid geometry, with the advantage for us that they do not require finiteness properties of the rings involved. The speaker first discusses affinoid rings, and then builds up the definition of the "adic spectrum", a geometric space associated to them defined in terms of their valuations.

Aim of the talk: introduce the category for *adic spaces*, a generalization of rigid geometry (giving us "analytic spaces" over a nonarchimedean base field) that's suitable for dealing with perfectoid rings. Two features of adic spaces:

1. Can treat general affinoid rings (no finiteness conditions).
2. The structure sheaf is defined on the underlying topology of a topological space (not on a Grothendieck topology). This reduces ambiguity in what we're doing - for instance open sets are determined on the level of points.

For the rest of the talk, fix a nonarchimedean base field k, i.e. k a field with a complete topology defined by a nontrivial rank-1 valuation $|\cdot| : k \to \mathbb{R}_{\geq 0}$. Examples: $\mathbb{Q}_p$, $\mathbb{F}_p((t))$, perfectoid fields.

Definition 1. An *f-adic ring* is a topological ring R such that R contains an open subring R_0 (the *ring of definition*) such that the topology on R_0 is adic with respect to a finitely-generated ideal of definition. An f-adic ring is R is a *Tate ring* if there exists a topologically nilpotent unit in R.

If R is a topological k-algebra, R is a Tate ring if the sets aR_0 for $a \in k^\times$ form a neighborhood basis of 0 for the topology on R. Examples: the usual Tate algebra from rigid geometry,

$$k\langle T_1, \ldots, T_n\rangle = k[T_1, \ldots, T_n]^\wedge$$

(where the completion is with respect to the ϖ-adic topology, for some $\varpi \in k^\times$ with $|\varpi| < 1$) is a Tate-k-algebra.

Definition 2. An element $a \in R$ in a f-adic ring is called *power-bounded* if the set $\{a^n : n \geq 0\}$ is a bounded subset of R. (In this situation, "boundedness" of a subset means that every neighborhood of 0 can be scaled to contain the subset). We let R° be the subring of power-bounded elements in R°.

Definition 3. An *affinoid algebra* is a pair (A, A^+) where A is an f-adic ring and $A^+ \subseteq A^\circ$ is an open and integrally closed subring of A°. An affinoid algebra (A, A^+) is *of finite type over* k if A is topologically of finite type over k (i.e. a quotient of a Tate algebra $k\langle T_1, \dots, T_n\rangle$) and $A^+ = A^\circ$.

From now on we let A denote an f-adic ring or a Tate ring.

Definition 4. A valuation on A is a map $v : A \to \Gamma \cup \{0\}$ where Γ is a totally ordered abelian group (which we write multiplicatively) satisfying:

- $v(0) = 0$ and $v(1) = 1$.
- $v(ab) = v(a)v(b)$
- $v(a + b) \leq \max\{v(a), v(b)\}$.

Here, we extend the order on Γ to $\Gamma \cup \{0\}$ by letting $0 < \gamma$ for all $\gamma \in \Gamma$, and extend multiplication by letting $0 \cdot ga = 0$.

Definition 5. Given a valuation on A as above, we let

$$\mathrm{supp}(v) = \{x \in A : v(x) = 0\},$$

which is a prime ideal, and we let Γ_v be the subgroup of Γ generated by the nonzero values $\{v(x) : x \in A, v(x) \neq 0\}$. We say two valuations v, v' on A are *equivalent* if, for all $a, b \in A$, we have $v(a) \leq v(b)$ iff $v'(a) \leq v'(b)$.

Two valuations v, v' are equivalent iff $\mathrm{supp}(v) = \mathrm{supp}(v')$ and the associated valuation rings on $\mathrm{Frac}(A/\mathrm{supp}(v))$ are the same.

Next, we want to define a notion of continuity for valuations. This is important because we want to, for instance, exclude the trivial valuation for $\mathbb{Q}_p$.

Definition 6. A valuation v is called *continuous* if, for all $\gamma \in \Gamma_v$, there exists an open neighborhoods U of 0 in A such that $v(x) < \gamma$ for all $x \in U$.

Definition 7. Let A be an f-adic ring. We define the *valuation spectrum*

$$\mathrm{Spv}(A) = \{\text{Equivalence classes of valuations } v : A \to \Gamma_v\}.$$

We further define the *continuous spectrum* as the subset

$$\mathrm{Cont}(A) = \{\text{Equivalence classes of continuous valuations } v : A \to \Gamma_v\}.$$

Finally, if (A, A^+) is affinoid we define a further subset of this

$$\mathrm{Spa}(A, A^+) = \{v \in \mathrm{Cont}(A) : v(x) \leq 1 \ \forall x \in A^+\}.$$

The spaces $\mathrm{Spa}(A, A^+)$ will be the analogues of affine sets for our definition of adic spaces. To make sense of this we need to first put a topology on it (and on $\mathrm{Cont}(A)$ and $\mathrm{Spv}(A)$ while we're at it); in each case we take the basis to be generated by the following sets for all $f, g \in A$:

$$S_{f,g} = \{v : v(f) \leq v(g) \neq 0\}.$$

This construction and the basic theory to follow is all due to Huber.

Theorem 8. *The spaces* $\mathrm{Spv}(A)$*,* $\mathrm{Cont}(A)$*, and* $\mathrm{Spa}(A, A^+)$ *are all spectral spaces (i.e. homeomorphic to the spectrum of some ring, given the Zariski topology).*

Given a ring (A, A^+) and a point $x = v \in \mathrm{Spa}(A, A^+)$, we want to view $f \in A$ as a function on this space and x as applying an absolute value for f. So we take the notation $|f(x)| = v(f)$.

If A is a Tate k-algebra, we can get a basis for the topology on $\mathrm{Spa}(A, A^+)$ by taking all rational subsets

$$U\left(\frac{f_1, \dots, f_n}{g}\right) = \{x \in \mathrm{Spa}(A, A^+) : |f_i(x)| \leq |g(x)|\}$$

for elements $f_1, \dots, f_n, g \in A$ with $(f_1, \dots, f_n) = A$. These subsets are all quasicompact (unlike the subsets in our original basis).

Example: Can compare this adic spectrum to the world of rigid spaces for the case that A is a Tate algebra of finite type over k. If $\mathfrak{m} \subseteq A$ is a maximal ideal, we get a valuation (technically, an equivalence class of them) by composing the projection $A \twoheadrightarrow A/\mathfrak{m}$ with the unique extension of the valuation on k to $A/\mathfrak{m}$ (which is a finite extension of k). This gives an inclusion

$$\mathrm{Sp}(A) = \{\mathfrak{m} \subseteq A \text{ a maximal ideal}\} \hookrightarrow \mathrm{Spa}(A, A^\circ).$$

The topology on $\mathrm{Spa}(A, A^\circ)$ recovers the Grothendieck topology on $\mathrm{Sp}(A)$ in the sense that $U \mapsto U \cap \mathrm{Sp}(A)$ gives a bijection between quasicompact opens of $\mathrm{Spa}(A, A^\circ)$ and quasicompact admissible opens in $\mathrm{Sp}(A)$. Similarly we have a bijection between coverings by quasicompact opens of $\mathrm{Spa}(A, A^\circ)$ and coverings by quasicompact admissible opens in $\mathrm{Sp}(A)$. So topologically we recover the rigid-analytic setup from the adic setup.

Of course, we don't just want a topological space, we want a structure (pre)sheaf. For an affinoid ring (A, A^+) we want to define a structure presheaf on $X = \mathrm{Spa}(A, A^+)$. We restrict to the case where A is Tate, so we can take rational subsets as our basis elements. Then, our construction mimics the one from classical rigid geometry: if $U = U\left(\frac{f_1, \dots, f_n}{g}\right)$ is a rational open subset, we take

$$\mathcal{O}_X(U) = A\left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle$$

where $A\langle f_1/g, \dots, f_n/g\rangle$ is the completion of $A[f_1/g, \dots, f_n/g] \subseteq A_g$ with respect to the topology generated by $aA_0[f_1/g, \dots, f_n/g]$ for $a \in k^\times$ (where we choose a ring of definition A_0, and the resulting topology is independent of the choice).

We also want to have a canonical integral structure on our structure sheaf. So we let

$$\mathcal{O}_X^+(U) = A\left\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g} \right\rangle^+$$

be the completion of the integral closure of $A^+[f_1/g, \ldots, f_n/g]$ with respect to the topology described above.

Proposition 9. *The canonical map* $\mathrm{Spa}(A\langle f_i/g\rangle, A\langle f_i, g\rangle^+)$ *to* $\mathrm{Spa}(A, A^+)$ *induced by the canonical map* $\varphi : (A, A^+) \to (A\langle f_i/g\rangle, A\langle f_i, g\rangle^+)$ *factors over* U *and induces a homeomorphism onto* U*. Moreover,* φ *is universal for maps* $(A, A^+) \to (B, B^+)$ *where* B *is complete and such that the induced map* $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ *factors over* U*.*

Since these sets U are a basis for a topology, what we've defined determines presheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$. In nicer situations you'd prove that these are sheaves, but adic spectra are not nice enough that this always holds. So we define

Definition 10. The spectrum $X = \mathrm{Spa}(A, A^+)$ as above is called an *affinoid adic space* if $\mathcal{O}_X$ is a sheaf.

In the situation of classical rigid geometry, we can of course actually verify that we have a sheaf.

Theorem 11. *If* A *is strongly Noetherian (i.e. if the ring* $A\langle T_1, \ldots, T_n\rangle$ *is Noetherian for all* n*) then* $\mathcal{O}_X$ *is a sheaf. (Note that Tate algebras of finite type over* k *are strongly Noetherian).*

To get general adic spaces, we glue affinoid ones. This gluing takes place in the category $\mathbf{V}$ consisting of triples

$$\big(X, \mathcal{O}_X, (v_x)_{x\in x}\big)$$

where X is a locally ringed space, $\mathcal{O}_X$ is a sheaf of complete topological rings, and v_x is a valuation on the stalk $\mathcal{O}_{X,x}$ for each $x \in X$. (Morphisms are the obvious ones). If (A, A^+) is an affinoid ring such that $\mathrm{Spa}(A, A^+)$ is an affinoid adic space, it determines such a triple by taking $X = \mathrm{Spa}(A, A^+)$, $\mathcal{O}_X$ the structure sheaf, and letting v_x be the canonical extension of the valuation defined by x to $\mathcal{O}_{X,x}$.

Definition 12. An *adic space* is an object of this category $\mathbf{V}$ which is locally isomorphic to an affinoid adic space.

Example: Projective line as an adic space. Start by considering the closed unit disc $X = \mathrm{Spa}(k\langle T\rangle, k^\circ\langle T\rangle)$, assuming k is complete and algebraically closed. Get $\mathbb{P}^1$ by gluing X and $X' = \mathrm{Spa}(k\langle T^{-1}\rangle, k^\circ\langle T^{-1}\rangle)$ in the obvious way. There's 5 types of points in X:
(1) Classical rigid points, corresponding to maximal ideals of $k\langle T\rangle$ and elements

of k°. (This is a rank-1 valuation).
(2),(3) Given $x \in k^\circ$ and a real number $0 < r \leq 1$ have a valuation $v_{x,r}$ given by

$$\sum a_i(T-x)^i \mapsto \sup\{|a_i|r^i\} = \sup\{|f(y)| : y \in D(x,r)\}.$$

where $D(x,r) = \{y \in k^\circ : |x-y| \leq r\}$. (The valuation only depends on the disc). We say it's type (2) if $r \in |k^\times|$ and type (3) if not. (These are also rank 1).
(4) If $D_0 \supseteq D_1 \supseteq D_2 \supseteq \cdots$ is a decreasing sequence of discs $D_i = D(x_i, r_i) \subseteq k^\circ$ with $\cap D_i = \emptyset$, then

$$f \mapsto \inf_i \sup_{y \in D_i} |f(y)|$$

is a valuation of type 1.
(5) Example: consider $v_{x,1}$ of type (2), which is actually independent of the point x. This has a specialization $\xi_{x,1}$ taking values in $R_{>0} \times \mathbb{Z}$ with the lexicographic order and a chosen generator $\gamma \in \mathbb{Z}$ with $\gamma < 1$. Then $\xi_{x,1}$ is given by $\sum a_i(T-x)^i \mapsto \sup\{(|a_i|, \gamma^i)\}$. These valuations $\xi_{x,1}$ depend only on $\{y : |x-y| < 1\}$, and form an $\mathbb{A}^1(\kappa)$ for κ the residue field. In $\mathbb{P}^1 \supseteq X$ there's an additional specialization of $v_{(x,1)}$ defined similarly but choosing $\gamma > 1$ (write it as $\xi_{(\infty,1)}$), and these specializations form a $\mathbb{P}^1(\kappa)$. For other points of type (2) with $r < 1$, we get a $\mathbb{P}^1(\kappa)$ in X. These are rank 2 valuations, and are closed points.

Points of type (1), (3), and (4) are closed points. Points of type (2) are not closed but have a specialization of type (5). Also, points of types (1)-(4) are those showing up in Berkovich spaces.

Can also see $\xi_{(\infty,1)}$ by changing the ring $k\langle T\rangle^+$. Namely, if we take $\overline{X} = X \cup \{\xi_{(\infty,1)}\} \subseteq \mathbb{P}^1$, we have that

$$\overline{X} = \mathrm{Spa}(k\langle T\rangle, k\langle T\rangle^+)$$

for $k\langle T\rangle^+$ the integral closure of $k^\circ + \varpi k^\circ\langle T\rangle$.

What are the value groups of each of these types of valuations? Type (1) obviously has value group $|k^\times|$. Can check type (2) also have value group $|k^\times|$, as do type (4), because those become type (4) if we replace k by its spherical completion. Type (3), on the other hand, are bigger than $|k^\times|$.

Remark: Define an *affinoid field* over k as a pair (K, K^+) where K is a complete nonarchimedean field and $K^+ \subseteq K^\circ$ is an open valuation subring. If $x \in \mathrm{Spa}(A, A^+)$ is a point then we get an affinoid field (K, K^+) with $K = \widehat{k(x)}$ and $K^+ = \widehat{k(x)^+}$ (completions of the residue field and its integral elements). Then points of $\mathrm{Spa}(A, A^+)$ correspond to equivalent classes of maps $(A, A^+) \to (K, K^+)$, similar to how points of an affine scheme can be viewed as equivalence classes of maps to fields. A point x is rank 1 iff this K^+ is equal to all of K°, and if y is a specialization of x then $K(x) = K(y)$ and $K(x)^+ \supseteq K(y)^+$. This is one reason why we really do need to allow a choice of K^+ in our affinoid fields, and not just restrict to the case $K^+ = K^\circ$ (and correspondingly for affinoid rings).