MSRI Hot Topics Workshop: Perfectoid Spaces and their Applications

Almost Ring Theory II: Perfectoid Rings - Bhargav Bhatt

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Summary: This lecture defines perfectoid algebras over a perfectoid field, and then aims to prove the "tilting equivalence" for categories of these objects. There are two major steps to this proof. The first is applying almost ring theory to construct an equivalence of perfectoid K-algebras with perfectoid $K^{\circ a}$-algebras. The second is using deformation theory (via the theory of the almost cotangent complex) to give an equivalence of perfectoid $K^{\circ a}$-algebras with perfectoid $K^{\circ a}/\pi$-algebras.

This talk will define perfectoid algebras and tilting (the affine version of perfectoid spaces). Start with a brief reminder of how tilting works for fields. Let K be a perfectoid field, which comes with the ring of integers K°. We then define

$$K^{\flat\circ} = \varprojlim_{\varphi} K^\circ/p;$$

this turns out to be an integral domain. We define $K^\flat$ as the fraction field of $K^{\flat\circ}$.

Theorem 1 (Fontaine-Wintenberger)**.** *$K^\flat$ is perfectoid, and the categories of finite étale algebras $K_{\text{fét}}$ and $K^\flat_{\text{fét}}$ are isomorphic. (In particular this implies the Galois groups are isomorphic).*

So we had K, the subring K°, and finally the quotient K°/p. We also have the tilt $K^\flat$, $K^{\flat\circ}$, and $K^{\flat\circ}/p$. The key point is that $K^\circ/p \cong K^{\flat\circ}/p$, and this is what lets us "build up" the tilt. Our goal is to extend this to a tilting equivalence between "perfectoid K-algebras" and "perfectoid $K^\flat$-algebras".

Part I: Tilting. Let K be a perfectoid field, and $\pi \neq 0$ an element of $\mathfrak{m}$ with $|p| \leq |\pi| < 1$.

Definition 2. A Banach K-algebra R is *perfectoid* if R° is open and bounded, and the Frobenius map $\varphi : R^\circ/p \to R^\circ/p$ is surjective. We define K-**Perf** as the category of perfectoid K-algebras with continuous maps.

Ultimately we want an equivalence of categories between K-**Perf** and $K^\flat$-**Perf**. To do this we need some intermediate objects (as in the field case above), this time involving almost ring theory.

Definition 3. A *perfectoid $K^{\circ a}$-algebra* is a flat $K^{\circ a}$-algebra R which is π-adically complete and such that φ gives an isomorphism $R/\pi^{1/p} \to R/\pi$. The category of these objects is $K^{\circ a}$-**Perf**.

Definition 4. A *perfectoid $(K^{\circ a}/\pi)$-algebra* is a flat $(K^{\circ a}/\pi)$-algebra such that φ induces an isomorphism $R/\pi^{1/p} \to R$. The category of these objects is $(K^{\circ a}/\pi)$-**Perf**.

Examples:
(1) Take $R = K\langle T^{1/p^\infty}\rangle$ in the Tate sense (i.e. take $\bigcup_n K^\circ[T^{1/p^n}]$, complete, and invert p). This is a perfectoid algebra over K, corresponding to the perfectoid affine line.
(2) There's also an integral version of the above construction, $K^\circ\langle T^{1/p^\infty}\rangle^a$.
(3) In characteristic p, and A is a Banach algebra over K with A° open and bounded, then A is perfectoid iff A is perfect.

Theorem 5 (Tilting equivalence)**.** *We have a chain of equivalences of categories*

$$\begin{array}{ccccc} K\text{-}\mathbf{Perf} & \xleftrightarrow{(A)} & K^{\circ a}\text{-}\mathbf{Perf} & \xleftrightarrow{(C)} & (K^{\circ a}/\pi)\text{-}\mathbf{Perf} \\ & & & & \updownarrow{=} \\ K^\flat\text{-}\mathbf{Perf} & \xleftrightarrow[(B)]{} & K^{\flat\circ a}\text{-}\mathbf{Perf} & \xleftrightarrow[(D)]{} & (K^{\flat\circ a}/\pi)\text{-}\mathbf{Perf} \end{array}$$

This theorem, along with any other unattributed ones in this talk, are due to Scholze. The vertical arrow in this theorem is the equality coming from the fact that the two mod-π almost algebras are isomorphic. The remaining equivalences come in two pairs.

Part II: K-Perf vs. $K^{\circ a}$-Perf. We start with a lemma:

Lemma 6. *Let M be a $K^{\circ a}$-module and N a K°-module. Then:*

1. *M is flat over $K^{\circ a}$ iff M_* is flat over K°.*
2. *N is flat over K° iff N^a is flat over $K^{\circ a}$ and we have $N^a_* = \{x \in N[1/\pi] : \forall \varepsilon \in \mathfrak{m}, \varepsilon x \in N\}$.*
3. *If M is flat over $K^{\circ a}$, then M is π-adically complete iff M_* is π-adically complete.*

Theorem 7. *We an equivalence of categories $K^{\circ a}$-**Perf** $\to K$-**Perf** given by $A \mapsto A_*[1/\pi]$ (and in the other direction by $R \mapsto R^{\circ a}$).*

We prove this by showing that the maps in both directions make sense.

Proposition 8. *If R is in K-**Perf** then $R^{\circ a} \in K^{\circ a}$-**Perf**.*

Proof. We have $\varphi : R^\circ/\pi \to R^\circ/\pi$ surjective by assumption. If $x \in R^\circ$ is such that $x^p \in \pi R^\circ$ then $x^p = \pi y$ for $y \in R^\circ$, and thus $(x/\pi^{1/p})^p = y \in R^\circ$. Thus $x/\pi^{1/p} \in R^\circ$, so $x \in \pi^{1/p} R^\circ$. This proves that φ gives an isomorphism $R/\pi^{1/p} \to R/\pi$. Moreover, $R^{\circ a}$ is clearly flat and is π-adically complete by the lemma. $\square$

Proposition 9. *If $A \in K^{\circ a}$-**Perf**, and we take $R = A_*[1/\pi]$ with the Banach algebra structure given by making A_* an open and bounded subring, then $A_* = R^\circ$ and $R \in K$-**Perf**.*

Proof. Start by proving $\varphi : A_*/\pi^{1/p} \to A_*/\pi$ is injective. This map is an almost isomorphism so this is almost injective. So, if $x \in A_*$ is such that $x^p \in \pi A_*$, then for all $\varepsilon \in \mathfrak{m}$ we have $\varepsilon x \in \pi^{1/p} A_*$. The lemma tells us that x is an almost element of $\pi^{1/p} A_*$, and the almost elements of this are just $\pi^{1/p} A_*$!

Next, suppose we have $x \in R$ with the property that $x^p \in A_*$; we want to prove that $x \in A_*$. We can write $y = \pi^{k/p} x \in A_*$ for some $k \geq 1$. Taking p-powers get $y^p = \pi^k x^p$, and this is in πA_* by assumption. By the previous paragraph, get $y \in \pi^{1/p} A_*$. But then this means we can write $y' = \pi^{(k-1)/p} x \in A_*$, and by induction we get it down to $x \in A_*$.

We now want to check that $A_*/\pi^{1/p} A_* \to A_*/\pi$ is surjective. We know it's almost surjective, so failure of surjectivity would be a quotient of A_*/π that's almost zero. This would have to factor through $A/\mathfrak{m}$. So it's enough to show $A_*/\pi^{1/p} \to A_*/\mathfrak{m}$ is surjective. So fix $x \in A_*$. Almost surjectivity gives that we can pick a y such that $\pi^{1/p} x = y^p \bmod \pi A_*$. Define $z = y/\pi^{1/p^2} \in R$; then $z^p = y^p/\pi^{1/p} = x \bmod \pi^{1-1/p} A_*$. Thus we conclude $z^p \in A_*$, and by the previous paragraph get $z \in A_*$, so x has a p-th root mod $\mathfrak{m}$.

There's a few more steps, but these are the most involved ones. $\square$

Part III: Review of cotangent complexes. This is a complex associated to a map of rings $A \to B$, defined originally by Quillen and André. For such a map, the cotangent complex is a complex $L_{B/A} \in D^{\leq 0}(B\text{-}\mathbf{mod})$ (which is an actual chain complex, but it's sufficient to think of it in the derived category). Some of its important properties are:

- $L_{B/A} = \Omega^1_{B/A}[0]$ if $A \to B$ is smooth.
- $L_{B/A}$ controls the deformation theory of f.

We'll be using the second property in a very special case. A baby case of what we want is as follows. Let $A = \mathbb{F}_p$ and B a perfect $\mathbb{F}_p$-algebra. Claim that the cotangent complex is zero. This follows because $\varphi : B \to B$ is an isomorphism, so the induced map $d\varphi : L_{B/A} \to L_{B/A}$ is an isomorphism by functoriality, but also $d\varphi = 0$ because (heuristically) we're differentiating p-powers in characteristic p. Thus there's no obstructions to and no choices involved in the deformation theory, and we get:

Corollary 10. *There exists a unique flat $\mathbb{Z}/p^n\mathbb{Z}$-algebra $W_n(B)$ (the ring of Witt vectors) lifting B.*

Part IV: Going from $K^{\circ a}$-Perf to $(K^{\circ a}/\pi)$-Perf, via deformation theory. Here everything is almost rings, but fortunately Gabber-Romero have developed a lot of the theory of cotangent complexes in the almost setting.

Fact: If $A \to B$ is a map of $K^{\circ a}$-algebras, then there's an almost cochain complex $L_{B/A} \in D(B\text{-}\mathbf{mod})$, which has the expected properties for deformation theory.

Lemma 11. *Let A be a $K^{\circ a}/\pi$-perfectoid algebra. Then $L_{A/(K^{\circ a}/\pi)} = 0$.*

This is proved by using that the relative Frobenius is an isomorphism, which reduces to the situation above where we had the absolute Frobenius for $\mathbb{F}_p$-algebras.

Then, the lemma gives that $(K^{\circ a}/\pi)$-**Perf** is equivalent to $(K^{\circ a}/\pi^n)$-**Perf** for any n (since the deformation theory is trivial), and taking inverse limits gives that it's equivalent to $K^{\circ a}$-**Perf**.

Explicitly, if A is a perfectoid $(K^{\flat\circ a}/t)$-algebra, then $A^\flat$ is $\varprojlim_\varphi A$ as a perfectoid $K^{\flat\circ a}$-algebra. (The inverse limit is done in the almost category; but we can think about constructions like it explicitly by pushing everything to the real world by using one of our functors M_* or $M_!$, then do constructions there, and apply the almost functor to get back).

Part V: Tilting étale covers. We want to show we have a similar chain of equivalences of categories of finite étale covers

$$\begin{tikzcd}
A_{\text{fét}} \arrow[r, leftrightarrow, "(A)"] & A^{\circ a}_{\text{fét}} \arrow[r, leftrightarrow, "(C)"] & (A^{\circ a}/\pi)_{\text{fét}} \arrow[d, leftrightarrow, "="] \\
A^{\flat}_{\text{fét}} \arrow[r, leftrightarrow, "(B)"'] & A^{\flat\circ a}_{\text{fét}} \arrow[r, leftrightarrow, "(D)"'] & (A^{\flat\circ a}/t)_{\text{fét}}
\end{tikzcd}$$

To make all of this work, need to make sure the condition of being finite étale is preserved under each link in our equivalences of categories of perfectoid algebra. Deformation theory tells us this is okay for (C) and (D). Equivalence (B) comes from the characteristic-p almost purity theorem from the end of an earlier lecture. Finally, equivalence (A) is the characteristic-0 almost purity theorem, which will be talked about later.