Adic Spaces II: Perfectoid Rings - Peter Scholze

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Summary: In this talk the speaker specializes the setup of adic spaces to considering "perfectoid affinoid algebras" and their associated adic spectra. The most important properties of these affinoid perfectoid spaces are discussed, in particular the tilting correspondence for spaces and the fact that the structure presheaves are actually sheaves. Finally, perfectoid spaces are defined as adic spaces that are locally affinoid perfectoid.

Fix a perfectoid base field K. Have ring of integers K° in K, which we'll also denote $\mathcal{O}_K$, and this has a maximal ideal $\mathfrak{m}$. From yesterday's talks we now that K has a *tilt* $K^\flat$ containing $\mathcal{O}_{K^\flat} = \mathcal{O}_K^\flat = \varprojlim \mathcal{O}_K/p$, with a maximal ideal $\mathfrak{m}^\flat$. We fix elements $0 \neq \varpi \in \mathfrak{m}$ and $\varpi^\flat \in \mathfrak{m}^\flat$ such that $\mathcal{O}_K/\varpi \cong \mathcal{O}_{K^\flat}/\varpi^\flat$.

Definition 1. An affinoid K-algebra (R, R^+) is called *perfectoid* if R is a perfectoid K-algebra.

Remarks:
(1) We must have $\mathfrak{m}R^\circ \subseteq R^+ \subseteq R^\circ$, so $R^+ \hookrightarrow R^\circ$ is an almost isomorphic.
(2) R^+ and R° carry a ϖ-adic topology and are ϖ-adically complete.

Proposition 2. *There is a tilting equivalence between perfectoid K-algebras (R, R^+) and perfectoid $K^\flat$-algebras (S, S^+), extending the tilting correspondence for perfectoid fields.*

Explicitly, this construction can be made by letting $R^\flat = \varprojlim R$ and $R^{\flat+} = \varprojlim R^+$ as multiplicative monoids (where the inverse limits are along the maps $x \mapsto x^p$). When we do this, we have $R^+/\varpi \cong R^{\flat+}/\varpi^\flat$. Can then define addition on these to make them algebras. One way to do this is as in Bhatt's talk yesterday, where we defined $R^{\flat\circ} = \varprojlim_\varphi R^\circ/\varpi$ as an algebra; there's a natural map $\varprojlim R^\circ \to \varprojlim R^\circ/\varpi$, and it turns out to be a multiplication-preserving continuous isomorphism. Also, writing $R^\flat$ as $\varprojlim R$ means we have a natural map $R^\flat \to R$ given by projection to the first factor; we denote this by $f \mapsto f^\sharp$.

Now that we have perfectoid affinoid algebras, we can take their adic spectra as in Hellmann's talk.

Proposition 3. *There exists a continuous map* $\mathrm{Spa}(R, R^+) \to \mathrm{Spa}(R^\flat, R^{\flat+})$ *denoted by* $x \mapsto x^\flat$, *such that for all* $f \in R^\flat$ *we have* $|f(x^\flat)| = |f^\sharp(x)|$.

This requires checking a few things (e.g. that defining $x^\flat$ by the formula actually gives a valuation); the only one that isn't straightforward is showing the strong triangle inequality for $x^\flat$. To prove this, fix $f, g \in R^\flat$. Rescaling by an element of K lets us assume without loss of generality that f, g are both in $R^{\flat+}$ but not both in $\varpi R^{\flat+}$. Then, we have $f^\sharp \bmod \varpi = f \bmod \varpi^\flat$ under the identification $R^+/\varpi \cong R^{\flat+}/\varpi^\flat$, so we can conclude $f^\sharp + g^\sharp \equiv (f+g)^\sharp \pmod{\varpi}$. So

$$|(f+g)(x^\sharp)|^{1/p^n} = |(f^{1/p^n} + g^{1/p^n})^\sharp(x)| \leq \max\{|\varpi|, |((f^\sharp)^{1/p^n} + (g^\sharp)^{1/p^n})(x)|\}$$

$$\leq \max\{|\varpi|, |f^\sharp(x)|^{1/p^n}, |g^\sharp(x)|^{1/p^n}\} = \max\{|f^\sharp(x)|, |g^\sharp(x)|\}^{1/p^n}$$

for sufficiently large n.

Theorem 4. *Let* (R, R^+) *be a perfectoid affinoid* K-*algebra with tilt* $(R^\flat, R^{\flat+})$.

1. *The map* $X = \mathrm{Spa}(R, R^+) \to \mathrm{Spa}(R^\flat, R^{\flat+}) = X^\flat$ *just defined is a homeomorphism preserving rational subsets.*

2. *The natural structure presheaves* $\mathcal{O}_X$, $\mathcal{O}_X^+$, $\mathcal{O}_{X^\flat}$, $\mathcal{O}_{X^\flat}^+$ *are all sheaves.*

3. *If* U *is a rational subset of* X, *then* $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ *is a perfectoid affinoid* K-*algebra with tilt* $(\mathcal{O}_{X_\flat}(U), \mathcal{O}_{X_\flat}^+(U))$.

4. *For all* $x \in X$, *the completed residue field* $\widehat{k(x)}$ *is a perfectoid field with tilt* $\widehat{k(x^\flat)}$.

Recall from Hellmann's talk that Huber showed that if R was strongly Noetherian, then the structure presheaf on $\mathrm{Spa}(R, R^+)$ was a sheaf. Here, we have no finiteness assumptions - instead we have the perfectoid condition, which is really a certain type of "bigness" assumption!

To prove part (1) of the theorem, we need an approximation lemma that if $f \in R$, there exists a $g \in R^\flat$ such that $f - g^\sharp$ is small in some sense; this will be discussed later in Caraiani's talk. For now, we talk about proving part (2), that the structure sheaves are sheaves. First we will show that if the characteristic of K is p, then $\mathcal{O}_X$ is a sheaf. We start by making the following definition in this setting:

Definition 5. We say a perfectoid affinoid (R, R^+) is *p-finite* if there exists a Tate-K-algebra (S, S^+) topologically of finite type such that R^+ is the ϖ-adic completion of $\varinjlim S^+$ and $R = R^+[1/\varpi]$.

Proposition 6. *In this situation, there is a homeomorphism preserving rational subsets*

$$X = \mathrm{Spa}(R, R^+) \cong \mathrm{Spa}(S, S^+) = Y.$$

Moreover, for any rational subset $U \subseteq X$, *the pair* $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ *is p-finite and is the perfection of* $(\mathcal{O}_Y(U), \mathcal{O}_Y^+(U))$.

This is straightforward to prove, and as a consequence we get the sheaf property for p-finite perfectoid affinoids.

Corollary 7. *Let X be as above and have a finite cover by rational subsets $U_i \subseteq X$. Then the complex*

$$0 \to R^+ \to \prod_i \mathcal{O}_X^+(U_i) \to \prod_{i,j} \mathcal{O}_X^+(U_i \cap U_j) \to \cdots$$

is almost exact.

Proof. We know that the complex

$$0 \to S \to \prod_i \mathcal{O}_Y(U_i) \to \prod_{i,j} \mathcal{O}_Y(U_i \cap U_j) \to \cdots$$

is exact by Tate's theorem. We can then apply Banach's open mapping theorem can conclude that all cohomology groups of the complex

$$0 \to S^+ \to \prod_i \mathcal{O}_Y^+(U_i) \to \prod_{i,j} \mathcal{O}_Y^+(U_i \cap U_j) \to \cdots$$

are killed by a power of ϖ. Now we pass to the perfection by taking the direct limit over Frobenius, and conclude all cohomology groups are killed by all ϖ^{1/p^n}'s and thus are almost zero. Completing gives the result. $\square$

We can extend this to the general case (still where K is characteristic p) via the following proposition.

Proposition 8. *Any perfectoid affinoid K-algebra (R, R^+) is a completed filtered direct limit of p-finite ones.*

We note that a filtered direct limit of almost exact sequences is almost exact.

Corollary 9. *The structure presheaves of affinoid perfectoid spaces in characteristic p are actually sheaves.*

From this we can prove the almost purity theorem. Let R be a perfectoid affinoid K-algebra. From Bhatt's talk, we had equivalences of categories

$$
\begin{array}{ccc}
R_{\text{fét}} & R_{\text{fét}}^{\circ a} \longleftrightarrow (R^{\circ a}/\pi)_{\text{fét}} \\
& \Big\updownarrow {\scriptstyle =} \\
R_{\text{fét}}^\flat \longleftrightarrow R_{\text{fét}}^{\flat \circ a} \longleftrightarrow (R^{\flat \circ a}/t)_{\text{fét}}
\end{array}
$$

Want to complete the chain by showing $R_{\text{fét}} \cong R_{\text{fét}}^{\circ a}$. We have an inclusion $R_{\text{fét}}^{\circ a} \hookrightarrow R_{\text{fét}}$, which we can check is fully faithful. Composing with the chain of equivalences of categories, get a functor $R_{\text{fét}}^\flat \hookrightarrow R_{\text{fét}}$, inverse to tilting.

Theorem 10 (Almost Purity)**.** *The functor $R^\flat_{\text{fét}} \hookrightarrow R_{\text{fét}}$ is an equivalence of categories. In other words, for any finite étale R-algebra S, we have that S is perfectoid and $S^{\circ a}/R^{\circ a}$ is finite étale.*

This was proven under some hypotheses by Faltings, trying to follow the proof of the Zariski-Nagata purity theorem, using a careful analysis of the integral structure. However, we can prove it in a different way.

Proof. Set up by picking $R^+ = R^\circ$, taking $X = \text{Spa}(R, R^+)$ and $X^\flat = \text{Spa}(R^\flat, R^{\flat+})$. Take any such S, and sheafify: for $U \subseteq X$ rational take $S(U) = S \otimes_R \mathcal{O}_X(U)$, and this gives a finite étale $\mathcal{O}_X(U)$-algebra.

Lemma: Fix $x \in X$. Then we have

$$\text{2-}\varinjlim_{x \in U} \mathcal{O}_X(U)_{\text{fét}} \cong \widehat{k(x)}_{\text{fét}}.$$

Proof of lemma:

$$\text{2-}\varinjlim_{x \in U} \mathcal{O}_X(U)_{\text{fét}} \cong \left(\varinjlim \mathcal{O}_X(U) \right)_{\text{fét}} = (\mathcal{O}_{X,x})_{\text{fét}}.$$

But $\varinjlim \mathcal{O}_X^+(U)$ is Henselian along ϖ, so we get

$$(\mathcal{O}_{X,x})_{\text{fét}} \cong \left((\mathcal{O}_{X,x}^+)^\wedge[1/\varpi] \right)_{\text{fét}}.$$

Finally, we identify what this last category is. We have an exact sequence

$$0 \to I \to \mathcal{O}_{X,x}^+ \to k(x)^+ \to 0$$

with I a K-vector space. If $f \in I$ then $|f| \leq |\varpi|$ on an open neighborhood of x, so f/ϖ is in the kernel of $\mathcal{O}_{X,x}^+ \to k(x)^+$, which is I. So ϖ-adic completion of $\mathcal{O}_{X,x}^+$ is equal to the completion of $k(x)^+$, which gives the equivalence in the statement of the lemma.

Corollary: Since we know that tilting preserves residue fields,

$$\text{2-}\varinjlim_{x \in U} \mathcal{O}_X(U)_{\text{fét}} \cong \widehat{k(x)}_{\text{fét}} \cong \widehat{k(x^\flat)}_{\text{fét}} \cong \text{2-}\varinjlim_{x \in U} \mathcal{O}_{X^\flat}(U)_{\text{fét}}.$$

So, locally $S(U)$ is in the image of the functor $\mathcal{O}_{X^\flat}(U)_{\text{fét}} \to \mathcal{O}_X(U)_{\text{fét}}$. Gluing gives the proof of the theorem we were after. $\square$

Finally, we define what a perfectoid space is in general.

Definition 11. A *perfectoid space* over K is an adic space over K that's locally isomorphic to $\text{Spa}(R, R^+)$ for (R, R^+) a perfectoid affinoid K-algebra.

Corollary 12. *The category of perfectoid spaces over K is equivalent to the category of perfectoid spaces over $K^\flat$ via a tilting equivalence $X \mapsto X^\flat$. This equivalence is such that $|X| = |X^\flat|$ and $\mathcal{O}_X$ tilts to $\mathcal{O}_{X^\flat}$ when evaluated on an affinoid perfectoid $U \subseteq X$.*