MSRI Hot Topics Workshop: Perfectoid Spaces and their Applications

Introduction to p-adic Comparison Theorems - Jean-Marc Fontaine

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Summary: The speaker discusses adic spaces and perfectoid spaces from another perspective that emphasizes the relationship with p-adic Hodge theory, and in particular the construction of period rings. We define two important classes of topological rings, "spectral rings" and "perfectoid rings". We then give a general construction of relative period rings that works in a relative setting.

Throughout the lecture, we fix a prime number p.

The key object we'll work with is the ring of Witt vectors $W(R)$ of an algebra R over $\mathbb{F}_p$. We'll start by describing a way to define this by "stupid multiplicative deformation theory". Our setup will be letting Λ be a ring and R a Λ-algebra. For $n > 0$, a Λ-*infinitesimal lifting* of order $\leq n-1$ is a pair (A, I) with A a Λ-algebra, I an ideal, $A/I \cong R$, $I^n = 0$. These form a category.

Now, consider triples (A, I, σ) where (A, I) is an infinitesimal thickening of order $\leq n-1$ and $\sigma : R \to A$ a multiplicative section. This also gives a category, and that there's an initial object in this category: take $\Lambda[R^\times]$ where here we let $R^\times$ be the multiplicative monoid of R. Define $\varepsilon : \Lambda[R^\times] \to R$ by $\sum \lambda_x[x] \mapsto \sum \lambda_x x$. The initial object is then $\Lambda[R^\times]/(\ker \varepsilon)^n$ which we denote $U_{n,\Lambda}(R)$.

We can then recover the Witt vectors from this setup as $W_n(R) = U_{n,\mathbb{Z}}(R) = U_{n,\mathbb{Z}_p}(R)$.

Next, we revisit the tilting correspondence discussed in the previous lectures. We start with some definitions. A *Banach ring* A is a topological ring containing a *pseudo-uniformizer* π (an invertible element that's topologically nilpotent) and an open subring $A_0 \ni \pi$ such that $A = A_0[1/\pi]$ and the natural map $A_0 \to \varprojlim A_0/\pi^n A_0$ is an isomorphism.

A *spectral ring* is a Banach ring A such that A° is bounded. (Recall that A° is the set of elements $a \in A$ that are "bounded", i.e. such that $\{a^n : n \in \mathbb{N}\}$ sits inside some $\pi^{-n}A_0$). If A is a Banach ring, A is a spectral ring iff there exists a power-multiplicative norm defining the topology. (Norms in general have to satisfy $|ab| \leq |a| \cdot |b|$ and $|1| = 1$; "power-multiplicative" means we require

$|a^n| = |a|^n$). If A has a power-multiplicative norm then $A^\circ = \{a : |a| \le 1\}$ is bounded; to prove the converse, you choose a pseudo-uniformizer π and a real number $0 < \rho < 1$, and define the norm $|a| = \rho^{v_\pi(a)}$ for

$$v_\pi(a) = \sup\{r/s : r, s \in \mathbb{Z} \text{ such that } a^s/\pi^r \in A^\circ\}.$$

Now, we can define a *perfectoid ring* as a spectral ring A such that there's a pseudo-uniformizer π with $p \in \pi^p A^\circ$ and with the Frobenius map $A^\circ/\pi^p A^\circ$ surjective. We can check that a perfectoid field (by our prior definition) is exactly a perfectoid ring which is a field and has a multiplicative norm.

If A is a spectral ring of characteristic p, then A is perfectoid iff it's perfect. If we take π to be a pseudo-uniformizer, we can form a Laurent series field $\mathbb{F}_p((\pi))$ in A, and if we take the radical and complete that gives a perfectoid field inside of A. So a perfectoid ring of positive characteristic always contains an embedded perfectoid field! On the other hand, if A has characteristic zero we can check that A is a Banach p-adic algebra, but we don't have a way to build a perfectoid field inside of it.

We now move on to tilting. If A is a perfectoid ring of characteristic zero, we want to define the *tilt* $A^\flat$. There's two ways we can approach defining this. One way is to pass to A°/π, define $A^{\flat\circ}$ as a ring as the inverse limit along Frobenius $\varprojlim A^\circ/\pi$, and take $A^\flat = A^{\flat\circ}[1/\varpi]$. A second way (which shows up a lot in classical p-adic Hodge theory) is to define $A^\flat$ directly as

$$A^\flat = \varprojlim_{x \mapsto x^p} A,$$

which gives us a multiplication operation but not addition. We define addition as follows; if $(x^{(n)})$ is an element of $A^\flat$ (so by definition it's a sequence with $x^{(n)} \in A$ and $(x^{(n+1)})^p = x^{(n)}$), we define $x + y$ by

$$(x+y)^{(n)} = \lim_{m \to \infty} (x^{(n+m)} + y^{(n+m)})^{p^m}.$$

Can check this makes sense, and is compatible with our other definition.

Now, take $R = A^\flat$. We have a surjective map $\theta_A : W(R^\circ) \to A^\circ$ defined by

$$\theta_A\left(\sum_{i=0}^{\infty} p^i [a_i]\right) = \sum_{i=0}^{\infty} p^i a_i^{(0)} = \sum_{i=0}^{\infty} p^i a_i^\sharp.$$

Then $\ker\theta_A$ is a primitive ideal of degree 1: i.e. it is a principal ideal that can be generated by $[\varpi] + p\eta$, where ϖ is the pseudo-uniformizer of R, $[\varpi]$ is its Teichmüller lift, and $\eta \in W(R^\circ)^\times$ is a unit.

So we get a functor from the category of perfectoid rings of characteristic zero to the category of perfectoid pairs (R, I) with R a perfectoid ring of characteristic p and I is a primitive ideal of degree 1 in $W(R^\circ)$. This functor is an equivalence of categories. To get the functor in the other way, you associate (R, I) to $A^\circ = W(R^\circ)/I$ and $A = A^\circ[1/p]$.

If you start with a perfectoid field K of characteristic zero, we know $K^\flat$ is a perfectoid field of characteristic p. If $R = F$ is a perfectoid field of characteristic p and I is any primitive ideal of degree 1 of $W(F^\circ)$, then our functor gives $(R, I)^\sharp$ a perfectoid field of characteristic zero, with tilt F. So we see that we can recover every perfectoid field of characteristic zero as an "untilt" $(R, I)^\sharp$.

Fontaine-Wintenberger theorem: Take K_0 a finite extension of $\mathbb{Q}_p$, L an algebraic extension of K_0 which is infinitely ramified and such that the Galois group of the Galois closure is a p-adic Lie group. Then the norm field is isomorphic to $k_L((t))$, and this proves the isomorphism of Galois groups. If you take K to be the completion of L, then K is a perfectoid field and $K^\flat$ is the completion of the radical of $k_L((t))$.

If K is a fixed perfectoid field of characteristic zero, have $\theta_K : W(K^{\flat\circ}) \to K^\circ$ and $\ker \mathcal{O}_K = (\xi)$, and then if A is a perfectoid k-algebra it corresponds to $(A^\flat, (\xi))$ which recovers the equivalence of categories of perfectoid algebras over K and over $K^\flat$.

Now, let k be a perfect field of characteristic zero, E an ultrametric field whose residue field is k. Then define

$$W_{E^\circ}(R) = E^\circ \widehat{\otimes}_{W(k)} W(R) = \varprojlim (E^\circ/\pi^n) \otimes_{W(k)} W(R).$$

There's a map $R \to W_{E^\circ}(R)$ by $x \mapsto 1\widehat{\otimes}[x] = [x]$, which is a multiplicative section of $W_{E^\circ}(R) \to R$. (If E has characteristic p then the Teichmüller lift is just $[x] = x$ and $R \subseteq W_{E^\circ}(R) = E_\circ \widehat{\otimes}_k R$).

If R is a perfectoid k-algebra (where k is just a subfield of R° with the discrete topology), can consider the ring $W_{E^\circ}(R^\circ)$. Then if we choose pseudo-uniformizers π of E and ϖ of R. We then define the ring

$$B_E^b(R) = W_{E^\circ}(R)\left[\frac{1}{\pi}, \frac{1}{[\varpi]}\right].$$

If E° is a DVR, we can choose π a uniformizer, and can check that $B_E^b(R)$ is the set of elements

$$\sum_{i \gg -\infty} [a_i]\pi^i$$

with $a_i \in R$ and the set $\{a_i\}$ bounded. In general, if $(a_n)_{n\in\mathbb{N}}$ is a sequence in R and $(\nu_n)_{n\in\mathbb{N}}$ with $\nu_n \in E$ and $\nu_n \to 0$, can form an element $\sum [a_n]\nu_n \in B_E^b(R)$. Can form any element of $B_E^b(R)$ this way, but the expression is not unique.

If we choose an absolute value on E and a power multiplicative norm on R, this defines a power multiplicative norm on $B_E^b(R)$. Individually, all of the norms on E are equivalent and all of the norms on R are equivalent, but the induced norms on $B_E^b(R)$ aren't necessarily equivalent if we vary both of the norms. So what we do is fix a power-multiplicative norm $|\cdot|$ on R.

Proposition 1. *For all $\rho \in \mathbb{R}$, there exists a unique power multiplicative norm $|\cdot|_\rho$ on $B_E^b(R)$ such that if $a \in R$ then $|[a]|_\rho = |a|$ and such that $|\pi|_\rho = \rho$.*

Now, choose a nonempty interval $I = [\rho_1, \rho_2]$ with $0 < \rho_1 \leq \rho_2 < 1$. Define

$$|\alpha|_I = \max\{|\alpha|_{\rho_1}, |\alpha|_{\rho_2}\} = \sup\{|\alpha|_\rho : \rho \in I\}.$$

This gives yet another a power-multiplicative norm, so we can complete with respect to it and get a spectral E-algebra $B_{E,I}(R)$. Then, if E is a perfectoid field, one can show that $B_{E,I}(R)$ is a perfectoid E-algebra. Moreover, the tilt $B_{E,I}(R)^\flat$ is $B_{E^\flat,I}(R)$.

Finally, can use this setup to define B_{dR} and B_{crys} in the general context. Let A be a perfectoid $\mathbb{Q}_p$-algebra. Define

$$B_{\mathrm{dR}}^+(A) = \varprojlim B_n(A)$$

for $B_n(A)$ is the infinitesimal thickening of degree $\leq n-1$ of A in the category of $\mathbb{Q}_p$-Banach algebras. If you want a nice proof of existence, you take $\theta : W(A^{\flat\circ}) \twoheadrightarrow A^\circ$, invert p to get $\theta : W(A^{\flat\circ})[1/p] \twoheadrightarrow A$, show the kernel is (ξ), and $B_n(A) = W(A^{\flat\circ})[1/p]/(\xi^n)$. To get $B_{\mathrm{dR}}(A)$ take $B_{\mathrm{dR}}^+(A)[1/\xi]$.

How about B_{crys}? Define $A_{\mathrm{crys}}^f(A)$ as the sub-$W(R^\circ)$-algebra of $W(R^\circ)[1/p]$ generated by the $\xi^m/m!$. Then $A_{\mathrm{crys}}(A)$ is the completion, $B_{\mathrm{crys}}^+(A) = A_{\mathrm{crys}}(A)[1/p]$, and then can get $B_{\mathrm{crys}}(A)$.