MSRI Hot Topics Workshop: Perfectoid Spaces and their Applications

# Adic Spaces III - Peter Scholze

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**Summary**: In this lecture the speaker discusses some additional topics and questions in the theory of adic spaces and perfectoid spaces. First of all, we define the étale topology for perfectoid spaces. We then move on to discussing some open questions in the general theory of adic spaces. One issue is finding a suitable subcategory of adic spaces that contains both perfectoid spaces and rigid analytic spaces, but avoids some of the pathologies that occur in general. Finally, we talk about how to describe generic fibers of formal schemes as adic spaces.

We start by discussing the étale topology for perfectoid spaces. Start with the remark that perfectoid rings (and more generally spectral rings from Fontaine's talk) are always reduced. Why is that? If $x \in R$ is nilpotent, then $\varpi^{-n}x \in R^\circ$ for all $n$, and thus $K \cdot x \subseteq R^\circ$. This is impossible because $R^\circ$ is bounded. So we can't have an infinitesimal lifting criterion in this setup.

**Proposition 1.** *For adic spaces of finite type over $K$, open embeddings and finite étale covers generate the étale site.*

We note that the analogue of this for schemes is *not* true!

**Definition 2.** If $f : Y \to X$ is a map of perfectoid spaces, we say $f$ is étale if it's locally of the form $Y \hookrightarrow Z \twoheadrightarrow W \hookrightarrow X$ with the middle map $X \to W$ a finite étale map and the other two are open embeddings. We use this definition to define the étale site $X_{\text{ét}}$.

We can check that this satisfies the usual properties (e.g. closed under composition, which isn't immediately obvious).

**Proposition 3.** *If $X$ has tilt $X^\flat$, then:*

1. $X_{\text{ét}} \cong X^\flat_{\text{ét}}$.
2. *If $X = \mathrm{Spa}(R, R^+)$ is affinoid perfectoid has $H^0(X_{\text{ét}}, \mathcal{O}_X^+) = R^+$ and $H^i(X_{\text{ét}}, \mathcal{O}_X^+)$ is almost zero for $i > 0$.*

The proof of (1) is formal from $|X| = |X^\flat|$ and the fact that the almost purity theorem tells us finite étale maps are the same in both setups. For (2), we skip the proof (though it's related to the proof that $\mathcal{O}_X$ was a sheaf for a perfectoid space). We remark that if $X$ affinoid is of finite type, then $H^i(X_{\text{ét}}, \mathcal{O}_X)$ is $R$ for $i = 0$ and is zero for $i > 0$, but $H^i(X_{\text{ét}}, \mathcal{O}_X^+)$ may contain huge torsion (i.e. submodules of the form $K/\mathcal{O}_K$). But this torsion gets killed by perfectoid covers.

A further remark: In Niziol's talks, we'll deal with the following setup. Suppose $K$ is algebraically closed of characteristic zero, $X/K$ proper smooth, and look at $R\Gamma(X_{\text{ét}}, \mathcal{O}_X^+)$. If you invert $p$, then you get

$$R\Gamma(X_{\text{ét}}, \mathcal{O}_X^+)[1/p] = R\Gamma(X_{\text{ét}}, \mathcal{O}_X),$$

the usual coherent cohomology (and same on the analytic site). But if you look at the torsion,

$$R\Gamma(X_{\text{ét}}, \mathcal{O}_X^+) \otimes^{\mathbb{L}}_{\mathcal{O}_K} \mathcal{O}_K/p^n$$

is almost isomorphic to étale cohomology

$$R\Gamma(X_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z}) \otimes^{\mathbb{L}}_{\mathbb{Z}/p\mathbb{Z}} \mathcal{O}_K/p^n.$$

So $R\Gamma(X_{\text{ét}}, \mathcal{O}_X^+)$ is some sort of bridge between coherent cohomology and étale cohomology.

Now we move on to some general comments about adic spaces. Main question: Is there a good category containing both rigid analytic varieties and perfectoid spaces? In some sense, adic spaces give a positive answer, but they're not as good as we might want. The problem is that the sheaf property is hard to verify.

Example of a Banach algebra where the sheaf property is not satisfied (due to Rost, Buzzard). Consider $R = \mathbb{Q}_p\langle pT, pT^{-1}\rangle$, the algebra of functions on a strip $p^{-1} \leq |T| \leq p$. Let $M$ be a Banach-$R$-module with a Banach-$\mathbb{Q}_p$-basis $p^{-n}T^{-n}$ and $p^{-n}T^n$ for $n \geq 0$. Let $R_1 = \mathbb{Q}_p\langle pT, T^{-1}\rangle$ and $R_2 = \mathbb{Q}_p\langle T, pT^{-1}\rangle$. Then:

**Proposition 4.** *The element $1 \in M$ dies in $M\widehat{\otimes}_R R_1$ and $M\widehat{\otimes}_R R_2$. (In fact, really $M\widehat{\otimes}_R R_i = 0$).*

*Proof.* Situation is symmetric so just look at $R_1$. Write $p^{-n} = (p^nT^{-n})T^n$, where we have $p^nT^{-n} \in M$ and $T^{-n} \in R$. Since $p^{-n}T^n$ is a basis element of $M$, $\|p^{-n}T^n\| \leq 1$ and similarly since $T^{-n}$ is a basis element of $R_1$ we have $\|T^n\| \leq 1$. Combining this and using the definition of the norm on the tensor product, we find $\|p^{-n}\| \leq 1$ for all $n$, so $\|1\| \leq p^{-n}$. So we must have $\|1\| = 0$ and thus $1 = 0$. □

So, $R \oplus M$ (where we think of $M$ as the "square-zero ideal") violates the sheaf property - there's a nonzero element which is locally zero. However, note that $R \oplus M$ is not spectral in Fontaine's sense, since it has nilpotents.

**Proposition 5.** *If $R$ is spectral, then for any cover $X = \mathrm{Spa}(R, R^+)$ by rational subsets $U_i$, then $R \hookrightarrow \prod \mathcal{O}_X(U_i)$. In fact, $R \hookrightarrow \prod_{x \in X} \widehat{k(x)}$.*

This follows from a theorem of Berkovich:

**Theorem 6.** *The pullback of the supremum norm on $\prod \widehat{k(x)}$ makes $R^\circ$ the norm $\leq 1$-subring.*

So if $R$ is spectral, the injectivity part of the sheaf property is automatic.
Questions:
(1) Are there counterexamples to the sheaf property for spectral rings? (Would guess so, but hard to construct).
(2) Is there a spectral ring $R$ and a rational subset $U \subseteq X = \mathrm{Spa}(R, R^+)$ such that $\mathcal{O}_X(U)$ is not spectral? (Again, would guess so).
(3) Is there a simultaneous counterexample to (1) and (2)?

Might hope that "spectral rings that stay spectral on rational subsets" would have the sheaf property be true; if so, this might be a good category to work with.

Another question: Is "perfectoid" a local property? Namely, if $K$ is a perfectoid field, $X = \mathrm{Spa}(R, R^+)$ a perfectoid space that's affinoid, is it necessarily true that $R$ is perfectoid? By definition this being perfectoid means that there's a cover by rational subsets that are perfectoid, but it doesn't necessarily mean the global algebra is perfectoid. As long as this question isn't answered have to distinguish between affinoid subsets and perfectoid affinoid subsets.

Another problem is inverse limits. Consider the setup where $(X_i)_{i \geq 0}$ is a tower of reduced adic spaces of finite type over $K$, with finite transition maps. Let $X$ be a perfectoid space with a compatible system of maps to the tower $f_i : X \to X_i$. In this setup say that $X$ is the *naive inverse limit* if all $X_i = \mathrm{Spa}(R_i, R_i^+)$ are affinoid and $X = \mathrm{Spa}(R, R^+)$ is perfectoid affinoid and $R^+$ is the $\varpi$-adic completion of $\varinjlim R_i^+$. Say $X \sim \varprojlim X_i$ ("similar") if this is satisfied locally (i.e. there exists some cover where it's true).

Remark: The category of affinoid rings does not admit filtered direct limits. If you have a system of $(R_i, R_i^+)$'s, the question is what topology to put on $\varinjlim R_i^+$? You might give it the direct limit topology, but then it will usually not be an affinoid ring. Need a stronger topology, but no canonical choice for what to do. If you restrict to spectral affinoid rings, this goes through because it gives you a $\varpi$-adic topology on $\varprojlim R_i^+$, so you can take $R^+$ the $\varpi$-adic completion and $R = R^+[1/\varpi]$.

**Proposition 7.** *If $X \sim \varprojlim_i X_i$ in the above sense, then $X = \varprojlim X_i$ in a suitable category of locally spectral adic spaces (where the inverse limit in question actually exists).*

Question: Does it make sense to develop the theory of spectral adic spaces? If the sheaf property would be satisfied in this generality it would be a good idea. But you still lose a lot of information - no non-reduced spaces, so you don't see tangent spaces in the usual $K[\varepsilon]/(\varepsilon^2)$-valued points way.

If you look at the isomorphism between the Lubin-Tate and Drinfeld towers, then in our notation above it has

$$\varprojlim_i \mathcal{M}_i \sim \mathcal{M}_\infty \cong \mathcal{N}_\infty \sim \varprojlim_i \mathcal{N}_i.$$

Is it true that $\Omega^1_{\mathcal{M}_0}|_{\mathcal{M}_\infty}$ is isomorphic to $\Omega^1_{\mathcal{N}_0}|_{\mathcal{N}_\infty}$? If it is, you can't check this via any infinitesimal liftings.

Finally, want to say something about generic fibers of formal schemes. Let $R$ be a ring, complete in some $I$-adic topology with $I$ finitely-generated. Can consider $\mathrm{Spa}(R,R)$ with a structure presheaf; we didn't define adic spaces in this generality but the theory can go through. Assume $R$ is a $\mathbb{Z}_p$-algebra with $p \in I$. Then $\mathrm{Spa}(R,R)$ lies over $\mathrm{Spa}(\mathbb{Z}_p,\mathbb{Z}_p)$, which has two points, the special point and the generic point $\eta = \mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)$. Define the generic fiber of $\mathrm{Spf}(R)$ as the fiber product

$$\mathrm{Spa}(R,R) \times_{\mathrm{Spa}(\mathbb{Z}_p,\mathbb{Z}_p)} \mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p),$$

which is an open subset of $\mathrm{Spa}(R,R)$.

Remark: In any reasonable setup, $\mathrm{Spf}(R) \mapsto \mathrm{Spa}(R,R)$ is a fully-faithful functor from affine formal schemes to adic spaces. So it makes sense to literally take the generic fiber of $\mathrm{Spa}(R,R)$.

Examples:
(1) If $R = \mathbb{Z}_p\langle T\rangle$, then

$$\mathrm{Spa}(R,R)_\eta = \mathrm{Spa}(R[p^{-1}],R) = \mathrm{Spa}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)$$

is the closed unit disc.
(2) If $R = \mathbb{Z}_p[[T]]$ (with $I = (p,T)$), then $\mathrm{Spa}(R,R)_\eta$ is the open unit disc, which is not affinoid.
(3) If $R = \mathcal{O}_K[[T^{1/p^\infty}]]$ (i.e. the $(p,T)$-adic completion of $\mathcal{O}_K[T^{1/p^\infty}]]$) for $K$ a perfectoid field, $\mathrm{Spa}(R,R)_\eta$ is a "perfectoid open unit ball".

**Proposition 8.** *Fix $f_1,\ldots,f_r \in I$ such that $I = (p,f_1,\ldots,f_k)$. Then*

$$\mathrm{Spa}(R,R)_\eta = \bigcup_{n\geq 1} \mathrm{Spa}\left(R\langle f_1^n/p,\ldots,f_k^n/p\rangle[1/p],...\right)$$

*(the $R^+$ term is complicated).*

**Proposition 9.** *Assume $R$ is a flat complete $\mathcal{O}_K$-algebra (with $K$ perfectoid) such that $R/p^{1/p} \cong R/p$ by Frobenius. Then $\mathrm{Spa}(R,R)_\eta$ is a perfectoid space over $K$.*