MSRI Hot Topics Workshop: Perfectoid Spaces and their Applications

The Weight-Monodromy Conjecture - Ana Caraiani

2:30pm February 18, 2014

Notes taken by Dan Collins (djcollin@math.princeton.edu)

Keywords: Weight-monodromy conjecture, Perfectoid spaces, Tilting equivalence.

Summary: The speaker summarizes Scholze's work on the weight-monodromy conjecture in characteristic zero, outlining a proof of the conjecture for hypersurfaces in projective space. We first review the perfectoid space theory that we'll need, and in particular how perfectoid projective space behaves under tilting. Then we move on to discussing what the weight-monodromy conjecture says, and what Deligne proved in characteristic p. Finally, we discuss how Scholze uses tilting to obtain cases of the weight-monodromy conjecture in characteristic zero by combining tilting and Deligne's result; a key technical point is establishing a certain approximation lemma.

We'll talk about Scholze's work on the weight-monodromy conjecture in characteristic zero, focusing on the case of the proof for hypersurfaces in projective space. This will involve comparing geometry and étale sites in characteristic zero to characteristic p.

We start by recalling the tilting equivalence. Let K be a perfectoid field (recall we have an element ϖ with $|p| \leq |\varpi| < 1$ and have $\varphi : K^\circ/\varpi \twoheadrightarrow K^\circ/\varpi$). This "tilts" to a perfectoid field $K^\flat$ of characteristic p, which can be constructed as

$$K^\flat = \varprojlim_{x \mapsto x^p} K.$$

There's a natural map $K^\flat \to K$ given by projection to the first coordinate, denoted $f \mapsto f^\sharp$; pick $\varpi^\flat$ with $(\varpi^\flat)^\sharp = \varpi$. This is continuous multiplicative homomorphism, but not additive. Further, we showed that there was a homeomorphism $\mathrm{Spa}(K, K^\circ) \cong \mathrm{Spa}(K^\flat, K^{\flat\circ})$. Moreover, there was an equivalence of categories between finite extensions of L/K and $L^\flat/K^\flat$, which recovers the Fontaine-Wintenberger isomorphism $\mathrm{Gal}(\overline{K}/K) \cong \mathrm{Gal}(\overline{K}^\flat/K^\flat)$.

More generally, have a tilting equivalence of perfectoid spaces over K (i.e. adic spaces locally isomorphic to affinoid perfectoids $\mathrm{Spa}(R, R^+)$ for a perfectoid K-algebra R). This takes X to its tilt $X^\flat$. If R is a perfectoid R-algebra we also have a projection-to-the-first-coordinate map

$$R^\flat = \varprojlim R \to R$$

denoted $f \mapsto f^\sharp$, which is a continuous multiplicative homomorphisms that is not additive, but induces an additive isomorphism on $R^+/\varpi \cong R^{\flat+}/\omega^\flat$. With this setup, if $X = \mathrm{Spa}(R, R^+)$ we have a homeomorphism $X \to X^\flat$ that preserves rational subsets, determined by $x \mapsto x^\flat$ where $|f(x^\flat)| = |f^\sharp(x)|$. Proving that this is a homeomorphism involves an approximation lemma we'll discuss later.

Moreover, we have structure sheaves $\mathcal{O}_X, \mathcal{O}_X^+$ on X that tilt to $\mathcal{O}_{X^\flat}, \mathcal{O}_{X^\flat}^+$ on $X^\flat$. We define étale morphisms for perfectoid spaces as being combinations of finite étale morphisms and open immersions; with this definition we get an isomorphism of étale sites $X_{\text{fét}} \cong X^\flat_{\text{fét}}$. The key point here is that we can do this locally, and for $x \in X$ we have

$$\widehat{\mathcal{O}_{X,x}^+}[\omega^{-1}] \cong \widehat{k(x)}.$$

The sheaf property is used crucially for gluing.

So we can compare a perfectoid space to its tilt on the level of topological spaces and étale sites. Later on when we're dealing with the weight-monodromy conjecture we want to be able to do compare things more generally for locally Noetherian adic spaces. The examples to keep in mind will be the adic projective space $(\mathbb{P}^n_K)^{\mathrm{ad}}$, constructed by gluing spaces

$$\mathrm{Spa}\big(K\langle T_1, \dots, T_n\rangle, K^\circ\langle T_1, \dots, T_n\rangle\big),$$

and also the perfectoid projective space $(\mathbb{P}^n_K)^{\mathrm{perf}}$ coming from gluing

$$\mathrm{Spa}\big(K\langle T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}\rangle, K^\circ\langle T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}\rangle\big).$$

Definition 1. Let X/K be a perfectoid space and X_i/K be a filtered inverse system of Noetherian adic spaces, equipped with a compatible system of maps $\varphi_i : X \to X_i$. We say that X is *similar* to $\varprojlim X_i$, denoted $X \sim \varprojlim X_i$, if the following two conditions are satisfied:

1. The maps on topological spaces $|X| \to |X_i|$ induce a homeomorphism $|X| \cong \varprojlim |X_i|$.
2. For any point $x \in X$, let x_i be the image in X_i; then we gt maps $k(x_i) \to k(x)$ on residue field, and we require that the induced map $\varinjlim k(x_i) \to k(x)$ has dense image.

Remark: If $Y \to X_i$ is an étale morphism of adic spaces, then can consider $Y \times_{X_i} X$; this is similar to

$$\varprojlim_{j \geq i} Y \times_{X_i} X_j.$$

The key example of this idea is the projective perfectoid space.

Theorem 2. *Take $(\mathbb{P}^n_K)^{\mathrm{perf}}$ over K and tilt to $K^\flat$.*

1. *$(\mathbb{P}^n_K)^{\mathrm{perf}}$ tilts to $(\mathbb{P}^n_{K^\flat})^{\mathrm{perf}}$.*
2. *$(\mathbb{P}^n_K)^{\mathrm{perf}} \sim \varprojlim_\varphi (\mathbb{P}^n_K)^{\mathrm{ad}}$, where φ is defined on coordinates by $[x_0 : \cdots : x_n] \mapsto [x_0^p : \cdots : x_n^p]$.*
3. *There are homeomorphisms of topological spaces $|(\mathbb{P}^n_{K^\flat})^{\mathrm{ad}}| \cong |(\mathbb{P}^n_{K^\flat})^{\mathrm{perf}}|$ (and this homeomorphic to $|(\mathbb{P}^n_K)^{\mathrm{perf}}| \cong \varprojlim_\varphi |(\mathbb{P}^n_K)^{\mathrm{ad}}|$ via tilting). If we consider this chain of isomorphisms and further project the last inverse limit to the first coordinate, the resulting map $\pi : |(\mathbb{P}^n_{K^\flat})^{\mathrm{ad}}| \to |(\mathbb{P}^n_K)^{\mathrm{ad}}|$ is $[x_0 : \cdots : x_n] \mapsto [x_0^\sharp : \cdots : x_n^\sharp]$.*
4. *There is an isomorphism of étale topoi $(\mathbb{P}^n_{K^\flat})^{\mathrm{ad},\sim}_{\text{ét}} \cong \varprojlim_\varphi (\mathbb{P}^n_K)^{\mathrm{ad},\sim}_{\text{ét}}$.*
5. *If we take $U \subseteq |(\mathbb{P}^n_K)^{\mathrm{ad}}|$ and consider $\pi^{-1}[V] \subseteq |(\mathbb{P}^n_{K^\flat})^{\mathrm{ad}}|$ under the map from (3), there is a commutative diagram of topoi*

$$\begin{tikzcd}
V^\sim_{\text{ét}} \arrow[r] \arrow[d] & (\mathbb{P}^n_{K^\flat})^{\mathrm{ad},\sim}_{\text{ét}} \arrow[d] \\
U^\sim_{\text{ét}} \arrow[r] & (\mathbb{P}^n_K)^{\mathrm{ad},\sim}_{\text{ét}}.
\end{tikzcd}$$

We can check (1) this on affinoid spaces; main computation we need is that

$$(K^\circ/\varpi)\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle \cong (K^{\flat\circ}/\varpi^\flat)\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle.$$

Similarly can check (2) on affinoid pieces. Part (3) follows from part (2) and the fact you're in characteristic p (since we have an inverse limit along purely inseparable maps that are homeomorphisms on the underlying topological spaces). For (4) we critically use the statement that $\varprojlim k(x_i)$ has dense image in $k(x)$ to descend étale morphisms.

Corollary 3. *We have an isomorphism*

$$H^i((\mathbb{P}^n_{K^\flat})^{\mathrm{ad}}_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z}) \cong H^i((\mathbb{P}^n_K)^{\mathrm{ad}}_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$$

if K (and thus $K^\flat$) is algebraically closed.

Using all of these ingredients, we will give a proof of the weight-monodromy conjecture for hypersurfaces in projective space. First, we recall what the weight-monodromy conjecture says. Our setup is that k is a local field (with uniformizer π and residue field $\mathbb{F}_q$) and X/k a proper smooth variety, then for $\ell \neq p$ the étale cohomology group $V = H^i(X_{\overline{k}}, \overline{\mathbb{Q}}_\ell)$ has an action of $\mathrm{Gal}(\overline{k}/k)$. This action is characterized by the action of Frob and a nilpotent operator.

The fact that we have a Frobenius action tells us we can decompose V as

$$V = \bigoplus_{j=0}^{2i} V_j$$

where on each V_j, the eigenvalues of Frob are Weil numbers of weight j (i.e. have absolute value $q^{j/2}$ for every complex embedding). In general the monodromy operator N takes V_j to V_{j-2}. If X has good reduction, the Weil conjectures tell us that only one weight occurs, so the monodromy operator is trivial. In general this doesn't happen, and the weight-monodromy conjecture tells us what should happen:

Conjecture 4 (Weight-Monodromy Conjecture, Deligne)**.** *If V is obtained from étale cohomology of a proper smooth variety X/k as above, then for any $0 \le j \le i$ the monodromy operator $N^j : V_{i+j} \to V_{i-j}$ is an isomorphism. (Equivalently, the monodromy filtration is the same as the weight filtration).*

In equal characteristic this is know.

Theorem 5 (Deligne)**.** *Let $C/\mathbb{F}_q$ be a curve, and $x \in C(\mathbb{F}_q)$ be a point such that k is the local field of C at x and $X \to C \setminus \{x\}$ is smooth, then $X_k = X \times_{C\setminus\{x\}} \operatorname{Spec} k$ satisfies the weight-monodromy conjecture.*

We now want to reduce the mixed-characteristic situation to this via our perfectoid space theory. Note that if we start with a local field k, we can take the perfectoid field $K = k(\varpi^{1/p^\infty})^\wedge$, and it's sufficient to work over this because passing to K doesn't kill any of the information we need for the weight-monodromy conjecture. Look at a smooth hypersurface $Y \subseteq \mathbb{P}^n_K$, and can pass to $Y^{\mathrm{ad}} \subseteq (\mathbb{P}^n_K)^{\mathrm{ad}}$.

Then, there's a comparison theorem due to Huber. Suppose $\widetilde{Y}$ is an open neighborhood of Y^{ad}. We then pull things back via $\pi : \mathbb{P}^n_{\mathbb{C}_p^\flat} \to \mathbb{P}^n_{\mathbb{C}_p}$. We know $\widetilde{Y}$ pulls back to an open neighborhood, but the hypersurface Y pulls back to some sort of fractal that we can't work with. Want to replace the fractal by an approximation Z that's algebraic over $\mathbb{C}_p^\flat$. The diagram we get would then be:

$$\begin{tikzcd}
\mathbb{P}^n_{\mathbb{C}_p^\flat} \arrow[r, "\pi"] \arrow[d] & \mathbb{P}^n_{\mathbb{C}_p} \arrow[d] \\
\pi^{-1}[Y] \arrow[r, "\pi"] \arrow[d] & Y \arrow[d] \\
Z_{\mathbb{C}_p^\flat} & Y_{\mathbb{C}_p}
\end{tikzcd}$$

To find this Z, we need an approximation lemma:

Lemma 6. *If Y is cut out by a homogeneous polynomial of degree d, there exists a homogeneous polynomial g in $\mathbb{C}_p^\flat\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle$ such that $|f(x)| \leq \varepsilon$ iff $|g^\sharp(x)| \leq \varepsilon$ for all $x \in (\mathbb{P}^n_{\mathbb{C}_p})^{\mathrm{ad}}$.*

If we have this lemma, you get a map

$$H^i(Y_{\mathbb{C}_p,\text{ét}}, \overline{\mathbb{Q}}_\ell) \to H^i(Z_{\mathbb{C}_p^\flat,\text{ét}}, \overline{\mathbb{Q}}_\ell)$$

that's $\mathrm{Gal}(\overline{K}/K) \cong \mathrm{Gal}(\overline{K}^\flat/K^\flat)$-equivariant and compatible with cup products. This will finish the proof since the right-hand side satisfies the weight-monodromy conjecture and we can verify the left-hand side arises as a direct summand.

The approximation lemma above is implied by the following one.

Lemma 7. *Let*

$$f \in R^\circ = K^\circ\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle$$

be homogeneous of degree d. Then for all $\varepsilon, c > 0$ there exists

$$g \in R^{\flat\circ} = K^{\flat\circ}\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle$$

such that

$$|f(x) - g^\sharp(x)| \leq |\varpi|^{1-\varepsilon} \max\{|\varpi|^c, |f(x)|\}.$$

To prove this you use induction on c. The base case follows from $R^\circ/\varpi \cong R^{\flat\circ}/\varpi^\flat$. Use some almost mathematics to conclude that if $|f(x)| \leq |\varpi|^c$ then $f - g \in |\varpi|^{1-\varepsilon+c}\mathcal{O}_X^+(U)$.

Finally, to get that $H^i(Y, \overline{\mathbb{Q}}_\ell)$ is a direct summand of $H^i(Z, \overline{\mathbb{Q}}_\ell)$, suffices to look at top degree $i = 2\dim Y$. This is either an isomorphism or zero. If it's zero, you go through our diagram above and get

$$H^i(\mathbb{P}^n_{\mathbb{C}_p^\flat}, \overline{\mathbb{Q}}_\ell) \to H^i(Z_{\mathbb{C}_p^\flat}, \overline{\mathbb{Q}}_\ell)$$

has to be zero, which can't happen. So the map in top degree is an isomorphism, and using compatibility with cup product and Poincaré duality gets that it's a direct summand in all degrees.