MSRI Hot Topics Workshop: Perfectoid Spaces and their Applications

Lubin-Tate Spaces I - Jared Weinstein

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Summary: In this talk the speaker introduces the Lubin-Tate spaces via the deformation theory of formal groups. Deformation theory of formal groups gives a tower of adic spaces, and by using the theory of perfectoid spaces one can interpret the inverse limit of this tower as an actual geometric object. By working in this infinite level setting, we can obtain a much nicer theory than exists at any finite level. Finally, we begin describing how to relate these constructions to p-adic Hodge theory.

A first motivation for the Lubin-Tate space: Let $X(Np^m)$ be the usual modular curve considered as a rigid space. The supersingular locus will be a disjoint union

$$\coprod_{X(N)(\overline{\mathbb{F}}_p)^{ss}} \mathcal{M}_m$$

where $\mathcal{M}_m$ is the m-th Lubin-Tate space.

We now discuss the Lubin-Tate deformation problem. Let k be a perfect field of characteristic p (in fact assume it's algebraically closed for simplicity). Let H_0 be a 1-dimensional formal group over k of height $n < \infty$ (which excludes the formal additive group). Then H_0 can be defined in terms of a formal addition law given by power series

$$X +_{H_0} Y = X + Y + \text{ higher-order terms.}$$

Also, it has a multiplication law $[p]_{H_0}(T) = f(T^{p^n})$ for a power series $f(T) = cT + \cdots$ with $c \neq 0$.

A more mature point of view is to look at H_0 as a formal scheme Spf $k[[T]]$, and in fact a group object in the category of $\mathbb{Z}_p$-formal schemes over k. For the purposes of this talk we'll say an "adic k-algebra" is a topological k-algebra R complete with respect to $I \subseteq R$. We then have $H_0(R) \cong \mathrm{Nil}(R) = \sqrt{I}$, where $\mathrm{Nil}(R)$ is the set of topologically nilpotent elements.

Now, we want to do deformation theory - we'll do this in the category $\mathbf{C}$ of complete Noetherian $W(k)$-algebras with residue field k (where $W(k)$ is the ring of Witt vectors). Then:

Theorem 1. *Take the functor* $\mathbf{C} \to \mathbf{Set}$ *defined by* $R \mapsto \{(H,\iota)\}/\sim$ *where* H/R *is a formal group,* $\iota : H_0 \to H \otimes_R k$ *is an isomorphism, and we go modulo isomorphisms of these objects. This functor is representable by a ring* A_0 *which is (non-canonically) isomorphic to* $W(k)[[u_1, \dots, u_{n-1}]]$.

Remark: Let $\mathcal{O}_D$ be the endomorphism algebra of H_0; this contains $\mathbb{Z}_p$ because H_0 is a $\mathbb{Z}_p$-module. (Called it $\mathcal{O}_D$ because if we take $D = \mathcal{O}_D \otimes \mathbb{Q}_p$ then this is a division algorithm of invariant i/n). Then $\mathcal{O}_D^\times$ acts on the functor and therefore on the representing object A_0. The induced action on $W(k)[[u_1, \dots, u_{n-1}]]$ is very mysterious; we don't know of any formula for the action if $n \geq 2$.

Starting with this A_0, Drinfeld defined a tower of rings

$$A_0 \to A_1 \to A_2 \to \cdots$$

where A_m classifies triples (H, ι, φ) over R, where φ is a Drinfeld level structure (which is a homomorphism $\varphi : (\mathbb{Z}/p^m\mathbb{Z})^{\oplus n} \to H[p^m]$ with some other conditions).

The example of $n = 1$ is local class field theory. In particular, in this situation $H_0 \cong \widehat{\mathbb{G}}_m$, the formal multiplicative group. Then $A_0 = W(k)$, which we'll call $\mathcal{O}_{K_0}$. More generally, $A_m = W(k)[\zeta_{p^m}]$, which we'll call $\mathcal{O}_{K_m}$. (Remark: for k algebraically closed, H_0 is determined up to isomorphism by the height n. For $n = 2$ we have $H_0 = \widehat{E}$ for E/k a supersingular elliptic curve; then $A_m = \widehat{\mathcal{O}}_{X(Np^m),x}$ for $x \in X(Np^m)(k)^{ss}$).

Now we pass to the generic fiber

$$\mathcal{M}^{(0)}_{H_0,m} = (\mathrm{Spf}\ A_m)^{\mathrm{ad}}_\eta.$$

Why the (0)? In our deformation problem required ι to be an isomorphism. If we instead allow it to be a quasi-isogeny of height j, you get a different deformation problem and resulting generic fiber $\mathcal{M}^{(j)}_{H_0,m}$. This is non-canonically isomorphic to $\mathcal{M}^{(0)}_{H_0,m}$. What is a quasi-isogeny of a formal group? An isogeny of formal groups is a power series that commutes with the group operation, and quasi-isogeny is something that is an isomorphism in the isogeny category.

Why do we do this? We want to define a tower with as big of a group acting on it as possible. Let $\mathcal{M}_{H_0,m}$ denote $\coprod_{j \in \mathbb{Z}} \mathcal{M}^{(j)}_m$. Before, $\mathcal{M}^{(0)}_m$ had an action of $\mathcal{O}_D^\times$, but $\mathcal{M}_{H_0,m}$ has an action of the full group $D^\times$. Now, we want to formulate an object at infinite level

$$\mathcal{M}_{H_0,\infty} = \varprojlim_m \mathcal{M}_{H_0,m};$$

this is the Lubin-Tate tower (which for now we just takes as a formal object). It has the action of $\mathrm{GL}_n(\mathbb{Q}_p) \times D^\times$, and the cohomology realizes the local Langlands correspondence! To study cohomology it's okay to study at the tower at finite level and take limits of cohomology groups. But we want to study the whole thing as a geometric object.

So consider the case $n = 1$. Then we know $\mathcal{M}^{(0)}_{H_0,m} = \mathrm{Spa}(K_m, \mathcal{O}_{K_m})$. Morally speaking, we want to have

$$\mathcal{M}^{(0)}_{H_0,\infty} = \mathrm{Spa}(K_\infty, \mathcal{O}_{K_\infty})$$

for $K_\infty = (\bigcup_m K_m)^\wedge$. But this is a perfectoid field!

What should $\mathcal{M}_{H_0,\infty}$ represent as a functor (for general n)? We had a moduli interpretation originally but then we passed to the generic fiber, which makes things a bit subtle. Anyway, for an affinoid K_0-algebra (R, R^+), the adic space $\mathcal{M}_{H_0,\infty}(R, R^+)$ naively should parametrize triples (G, ι, φ) up to isogeny, with G/R^+ a formal group, $\iota : H_0 \otimes_k R^+/p \to G \to_{R^+} R^+/p$ a quasi-isogeny, and $\varphi : \mathbb{Q}_p^n \cong VG$ a level structure, where VG is the rational Tate module of G, $VG = TG \otimes \mathbb{Q}_p$ for $TG = \varprojlim_m G[p^m](R^+)$.

Why is this naive? Well, first of all R^+ is not necessarily p-adically complete, and we don't want to talk about formal groups over non-complete things. Secondly, we want the representing functor to be a sheaf; what we've defined is a presheaf, and we sheafify for the topology defined by rational subsets (which is the usual topology on $\mathrm{Spa}(R, R^+)$).

Return to the case $n = 1$. When we take the connected ting $\mathcal{M}^{(0)}_{H_0,\infty}$ and evaluate it at (R, R^+) we get $\mathrm{Hom}_{\mathcal{O}_{K_0}}(\mathcal{O}_{K_\infty}, R^+)$. This essentially corresponds to choosing where to put the roots of unity. So it's equal to the Tate module $T\mu_{p^\infty}(R^+)^{\mathrm{prim}}$ (where $T\mu_{p^\infty}(R^+)$ is a rank-1 module and the "prim" means we're restricting to elements that give a basis). Similarly, $\mathcal{M}_{H_0,\infty}(R, R^+)$ is $V\mu_{p^\infty}(R^+) \setminus \{0\}$.

Our next topic is formal vector spaces. Let R be a ring in which p is topologically nilpotent and H/R is a formal group that's p-divisible. Define $\widetilde{H}$ as the inverse limit $\varprojlim H$ where the transition maps are multiplication-by-p, where we work in the category of formal schemes. When we take this inverse limit we get a $\mathbb{Z}_p$-module, but moreover the action of p is invertible so we get a $\mathbb{Q}_p$-vector space object in the category of formal schemes. Call it a formal vector space.

Orienting example: If we consider the abstract group $\mathbb{Q}_p/\mathbb{Z}_p$, doing this same inverse limit process is $\mathbb{Q}_p$.

Proposition 2. *Consider our original formal group H_0/k. The formal vector space $\widetilde{H}_0$ is representable, and isomorphic to* Spf $k[[T^{1/p^\infty}]]$.

Proposition 3. *Suppose H/R is a p-divisible formal group. Get a $\mathbb{Q}_p$-linear homomorphism $\widetilde{H}(R) \to \widetilde{H}(R/p)$, which is an isomorphism.*

Combining these two propositions we can see:

Proposition 4. *Let $H/\mathcal{O}_{K_0}$ be a lift of H_0 (chosen arbitrarily). Then $\widetilde{H} \cong$* Spf $\mathcal{O}_{K_0}[[T^{1/p^\infty}]]$*; in particular it's independent of the choice of lift H.*

Corollary 5. *Let K/K_0 be a perfectoid field (possibly $K = \mathbb{C}_p$), and set $\eta =$* $\mathrm{Spa}(K, \mathcal{O}_K)$*. Then $\widetilde{H}^{\mathrm{ad}}_\eta$ is a perfectoid space.*

The underlying object of $\widetilde{H}^{\mathrm{ad}}_\eta$ is a 1-dimensional perfectoid unit disc, and it is a $\mathbb{Q}_p$-vector space object in the category of perfectoid spaces. Also, since the division algebra D acts by isogenies on H_0, it acts by isomorphisms on $\widetilde{H}_0$, and these actions lift canonically to give an action of D on $\widetilde{H}$.

Back to the context of the Lubin-Tate space. Let (R, R^+) be a perfectoid K-algebra and let (G, ι, φ) be a point in $\mathcal{M}_{H_0,\infty}(R, R^+)$. Starting with

$$\varphi : \mathbb{Q}_p^n \cong VG(R^+) = \mathbb{Q}_p \otimes \varprojlim G[p^m](R^+).$$

But, $\varprojlim G[p^m](R^+)$ sits inside of $\varprojlim G(R^+) = \widetilde{G}(R^+)$. We've seen that this is isomorphic to $G(R^+/p)$, and via ι^{-1} this is isomorphic to $\widetilde{H}_0(R^+/p) \cong \widetilde{H}(R^+)$. This determines a morphism $\mathcal{M}_{H_0,\infty} \to (\widetilde{H}^{\mathrm{ad}}_\eta)^n$. This morphism does not appear at finite levels, only at infinite level.

The next step is to connect to p-adic Hodge theory. Notation as before; H_0/k a formal group that's p-divisible, so it has a Dieudonné module $M(H_0)$ (which is a $W(k)$-module with endomorphisms F, V with $FV = p$). Let R be a k-algebra; we say R is "f-semiperfect" if $R = S/I$ with S perfect and I finitely generated. (Example: $\mathcal{O}_{\mathbb{C}_p}/p \cong \mathcal{O}_{\mathbb{C}_p^\flat}/p^\flat$).

Theorem 6 (Scholze-Weinstein)**.** *If R is f-semiperfect then*

$$\widetilde{H}_0(R) \cong \mathrm{Hom}_{F,\varphi}(M(H_0), B^+_{\mathrm{crys}}(R)) \cong B^+_{\mathrm{crys}}(R)^{\varphi^n = p}.$$

As a consequence, get following fact about formal linear algebra. If $\bigwedge^r M(H_0)$ is the r-th exterior power of the Dieudonné module, it's a Dieudonné module again, of some object we call $\bigwedge^r H_0$. By the theorem,

$$\widetilde{H}_0(R)^{\oplus r} \cong \mathrm{Hom}(M(H_0)^r, B^+_{\mathrm{crys}}(R))$$

and we can map this to

$$\mathrm{Hom}(\Lambda^r M(H_0), B^+_{\mathrm{crys}}(R)) \cong \widetilde{\Lambda^r H_0}(R).$$

So we get a $\mathbb{Q}_p$-alternating map $\widetilde{H}_0(R)^r \to \widetilde{\Lambda^r H_0}(R)$.

Now, recall we had a morphism $\mathcal{M}_{H_0,\infty} \to (\widetilde{H}^{\mathrm{ad}}_\eta)^n$. We also have $\mathcal{M}_{\Lambda^n H_0,\infty} \to \widetilde{\Lambda^n H}^{\mathrm{ad}}_\eta$. The discussion in the previous paragraph lets us induce a morphism $\det : (\widetilde{H}^{\mathrm{ad}}_\eta)^n \to \widetilde{\Lambda^n H}^{\mathrm{ad}}_\eta$. Then:

Theorem 7. *There is a Cartesian diagram*

$$\begin{tikzcd}
\mathcal{M}_{H_0,\infty} \arrow[r] \arrow[d, dotted] & (\widetilde{H}^{\mathrm{ad}}_\eta)^n \arrow[d, "\det"] \\
\mathcal{M}_{\Lambda^n H_0,\infty} \arrow[r] & \widetilde{\Lambda^n H}^{\mathrm{ad}}_\eta .
\end{tikzcd}$$

Moreover all objects here carry an action of $\mathrm{GL}_n(\mathbb{Q}_p) \times D^\times$ *that gives an action of* $\mathbb{Q}_p^\times$ *via* $\det \cdot N^{-1}$.

In fact, since $\mathcal{M}_{H_0,\infty}$ is the only object in this diagram we haven't fully defined, we can take this as a *definition* of this object as a fiber product (since those exist in the category of perfectoid spaces).