MSRI Hot Topics Workshop: Perfectoid Spaces and their Applications

The Fargues-Fontaine Curve - Laurent Fargues

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Summary: This lecture covers the basic theory of the Fargues-Fontaine curve. This is an adic curve (with an algebraic construction) that parametrizes all characteristic-zero "untilts" of a fixed characteristic-p perfectoid field. Using this curve we can thus make sense of trying to functorially untilt perfectoid algebras. We go over the construction of this curve, and using the construction study how we can explicitly describe it.

Suppose that $K/\mathbb{Q}_p$ is a perfectoid field; we know it has a tilt $K^\flat$ and there's an equivalence of perfectoid algebras over these. But what if we start with a perfectoid field F of characteristic p? Then there's no canonical choice of K of characteristic zero with $K^\flat = F$. In fact, if we fix K and look at the sets of pairs (K, ι) with K perfectoid of characteristic zero and $\iota : F \cong K^\flat$ and go modulo isomorphism and the Frobenius action (on F and thus ι) we have

$$\{(K,\iota)\}/\sim \times \mathrm{Frob}^{\mathbb{Z}} \cong |X_F|^{\deg 1}$$

where X_F is an "algebraic" curve. Our goal here is to construct X_F^{ad}.

More explicitly, we want to build functors

$$F\text{-}\mathbf{Perf} \to \mathbf{Adic}_{\mathbb{Q}_p}/X_F^{\mathrm{ad}} \to k(x)\text{-}\mathbf{Perf}$$

(where the composite is Scholze's equivalence, and the second map depends on a choice of $x \in |X_F| = |X_F^{\mathrm{ad}}|^{\deg 1}$) that gives a canonical way to lift perfectoid fields to characteristic zero.

We take the following setup: $\mathbb{F}_q$ is a finite field, $F/\mathbb{F}_q$ is perfectoid, E is non-archimedean with a chosen element ϖ_E such that $|p| \le |\omega_E| \le 1$. For $\rho \in (0,1)$ we can define an absolute value $|\cdot|_\rho$ on E with $|\varpi_E|_\rho \in \rho$. Now, if A is a perfectoid F-algebra with a spectral norm $|\cdot|$, for any closed interval $I \subseteq (0,1)$, Fontaine's talk constructed

$$B_{A,E,I} = B_{E,I}(A)$$

a perfectoid Banach E-algebra, which satisfies the following properties we need:

- If E'/E is an extension, $B_{A,E,I}\widehat{\otimes}_E E' = B_{A,E',I}$.

- If E is perfectoid, $B^\flat_{A,E,I} = B_{A,E^\flat,I}$.

Further, if A is a perfectoid affinoid algebra (so comes with data of $A^+ \subseteq A^\circ$), can construct $B^+_{A,E,I} \subseteq B^\circ_{A,E,I}$.

Definition 1. Define an adic space $Y_{A,E,I}$ as $\mathrm{Spa}(B_{A,E,I}, B^+_{A,E,I})$.

If $[\rho_1,\rho_2] = I \subseteq I'$ is a subinterval and moreover $\rho_1,\rho_2 \in |F^\times|$ (with $\rho_1 = |a|$ and $\rho_2 = |b|$, say), then we can recover $B_I = B_{A,E,I}$ from $B_{I'}$ as

$$B_I = B_{I'}\left\langle \frac{[a]}{\varpi_E}, \frac{\varpi_E}{[b]} \right\rangle.$$

So, if we have a containment of intervals $I \subseteq I'$ of this form, then $Y_I \subseteq Y_{I'}$ is a rational open.

Definition 2. Define a perfectoid space over E by

$$Y_{A,E} = \varprojlim_{I \subseteq (0,1)} Y_{A,E,I}.$$

If E is perfectoid then $Y^\flat_{A,E} = Y_{A,E^\flat}$.

Example: Suppose E has characteristic p. Then $Y_{A,E}$ sits over $\mathrm{Spa}(E)$ (by definition) but also over $\mathrm{Spa}(A)$. This is because $B_{A,E,I}$ is an A-algebra, as there was a Teichmüller character $[\cdot] : A \to B_{A,E,I}$ which is additive in characteristic p. Then we simply have $Y_{A,E} = \mathrm{Spa}(A) \otimes_{\mathrm{Spa}(F)} Y_{F,E}$. So characteristic p is quite simple.

Proposition 3. *Take elements $f_1,\ldots,f_n, g \in A$ with $(f_1,\ldots,f_n,g) = A$ and consider the rational subset defined by them. Then B_I is generated by the Teichmüller lifts $[f_1],\ldots,[f_n],[g]$; i.e.*

$$B_I = [g]B_I + \sum_i [f_i]B_I.$$

Moreover, we have

$$B_{A\langle\frac{f_1,\ldots,f_n}{g}\rangle,E,I} = B_{A,E,I}\langle\tfrac{[f_1],\ldots,[f_n]}{[g]}\rangle.$$

Using this localization property, can glue the functor $A \mapsto Y_{A,E}$ to a functor from perfectoid spaces over F to pre-perfectoid spaces over E that are fibered over $Y_{F,E}$; the resulting functor is $\mathfrak{Z} \mapsto Y_{\mathfrak{Z},E}$.

Now we need to talk about the Frobenius. If $\rho \in (0,1)$, set $\varphi(\rho) = \rho^q$. Then have Frob_q on A, which is the Frobenius isomorphism $\varphi : B_{A,E,I} \to B_{A,E,\varphi(I)}$ of E-Banach algebras. From Fontaine's talk, elements of $B_{A,E,I}$ can be written as $\sum_m [\chi_m]\lambda_m$ with $\chi_m \in A$ and $\lambda_m \in E$; the Frobenius is given by

$$\varphi\left(\sum_m [\chi_m]\lambda_m\right) = \sum_m [\chi_m^q]\lambda_m.$$

Note we need to have arithmetic Frobenius here, not geometric; if E is discrete with uniformizer π_E, can write elements as $\sum_{m\gg-\infty}[\chi_m]\pi_E^m$, and Frobenius takes it to $\sum_{m\gg-\infty}[\chi_m^q]\pi_E^m$.

We then return to our space $Y_{\mathfrak{Z},E} \to \mathrm{Spa}(E)$, where $\mathfrak{Z}$ is a perfectoid space over F. The Frobenius φ induces an automorphism of $Y_{\mathfrak{Z},E}$ that sends something of radius ρ to something of radius $\rho^{1/q}$ (which is bigger because $\rho \in (0,1)$).

Definition 4. For a F-perfectoid space $\mathcal{Z}$, we set $X^{\mathrm{ad}}_{\mathfrak{Z},E}$ to be the quotient $\varphi^{\mathbb{Z}}\backslash Y_{\mathfrak{Z},E}$. This makes sense because φ acts discontinuously, and the result is a pre-perfectoid space over E. This gives the functor we wanted $\mathbf{Perf}_F \to \mathbf{PrePerf}_E/X^{\mathrm{ad}}_{F,E}$. (The latter category is just the category of pre-perfectoid spaces over E together with morphisms to $X^{\mathrm{ad}}_{F,E}$).

Remark: If $\mathfrak{Z}$ is a F-perfectoid space, "$\mathrm{Frob}^{\mathbb{Z}}\backslash\mathfrak{Z}$" does not exist. But what we did was take $\mathfrak{Z} \times_{\mathrm{Spa}F} Y_{F,E}$, and now the quotient we want does exist.

What is this space concretely? Consider the case where E is a DVR with uniformizer π_E and residue field $\mathbb{F}_q$. If E has characteristic p, so $E = \mathbb{F}_q((\pi_E))$, then $Y_{F,E}$ is the punctured disc $\mathbb{D}_F^*$ which lies over Spa F and is locally of finite type. It also lies over the punctured disc $\mathbb{D}^*_{\mathbb{F}_q} = \mathrm{Spa}\ \mathbb{F}_q((\pi_E))$, but this structure map is not of finite type. Now, if we go modulo powers of Frobenius and consider $\varphi^{\mathbb{Z}}\backslash\mathbb{D}_F^*$, this no longer lies over F (since $\varphi^{\mathbb{Z}}\backslash\mathrm{Spa}\ F$ doesn't make sense), but it does still lie over Spa $\mathbb{F}_q((\pi_E))$ since that was Frobenius-invariant. The action of φ on $\mathbb{D}_F^*$ is as follows: a function on $\mathbb{D}_F^*$ is of the form $\sum a_m\pi_E^m$, and φ takes this to $\sum a_m^q\pi_E^m$. (Remark: In the case of $E = \mathbb{F}_q((\pi_E^{1/p^\infty}))$, then $Y_{F,E} = \mathbb{D}_F^{*1/p^\infty}$).

Now consider the case where E is of characteristic zero, in particular take $E/\mathbb{Q}_p$ a finite extension with residue field $k_E = \mathbb{F}_q$. Take $\mathcal{LT}$ to be the Lubin-Tate group law over $\mathcal{O}_E$. Let E_∞ be the extension of E obtained by adjoining the torsion points of $\mathcal{LT}$ (in a fixed algebraic closure) and completing. This gives us a perfectoid field. Now, let $\pi_E^\flat = (\pi_E^{\flat(m)})$ be a generator of the π_E-adic Tate module $T_{\pi_E}(\mathcal{LT})$. Then we have

$$[\pi_E]_{\mathcal{LT}}(\pi_E^{\flat(m+1)}) = \pi_E^{\flat(m)}.$$

When we reduce mod π_E, find that $\mathrm{Frob}_q(\pi_E^{\flat(m+1)}) = \pi_E^{\flat(m)}$. So $\pi_E^\flat \in E_\infty^\flat = \mathbb{F}_q((\pi_E^{\flat 1/p^\infty}))$.

Then, we have the Lubin-Tate character $\chi : \mathrm{Gal}(E_\infty/E) \to \mathcal{O}_E^\times$ which satisfies

$$(\pi_e^\flat)^\sigma = [\chi(\sigma)]_{\mathcal{LT}}(\pi_E^\flat).$$

Using this, we check the following. Take $\mathcal{G}$ to be the Lubin-Tate group over π_q, and then as in Weinstein's talk we can take $\widetilde{\mathcal{G}} = \varprojlim \mathcal{G}$, which is a formal E-vector space. Take $\mathcal{E} = (\widetilde{\mathcal{G}}\widehat{\otimes}_{\mathbb{F}_q}\mathcal{O}_F)_\eta$, which is an E-Banach space. We can then check that $Y_{F,E_\infty^\flat} \cong \mathcal{E}\setminus\{0\}$, with the Galois action of $\mathrm{Gal}(E_\infty/E)$ on $Y_{F,E_\infty^\flat}$ corresponding to the action of $\mathcal{O}_E^\times$ on $\mathcal{E}\setminus\{0\}$ via the character χ.

Back to our adic curve, we have

$$|X_{F,E}^{\mathrm{ad}}| = \mathrm{Gal}(E_\infty/E)\backslash|X_{F,E_\infty}| = \mathrm{Gal}(E_\infty/E)\backslash|X_{F,E^\flat{}_\infty}| \cong E^\times\backslash|\mathcal{E}\setminus\{0\}|.$$

This gives a description of the adic curve as a topological space.

Consider the case of $F = \mathbb{F}_q((\pi_F^{1/p^\infty}))$ and any E. Then can check that (where $T = [\pi_F]$) we have:

$$Y_{F,E} \cong \mathbb{D}_E^{*1/p^\infty} = \{T : 0 \le |T| \le 1\} \subseteq \mathrm{Spa}\ E\langle T^{1/p^\infty}\rangle.$$

But there's two radius functions floating around. For $r \in (0,\infty)$, the radius q^{-r} in $Y_{F,E}$ corresponds to radius $q^{-1/r}$ in the punctured disc. What's happening here? Consider the simpler case of $E = \mathbb{F}_q((\pi_E))$ and $F = \mathbb{F}_q((\pi_F))$. Look at $\mathbb{D}_E^*$ with variable π_F; this is the same as $\mathbb{D}_F^*$ with variable π_E. There is a canonical isomorphism $\mathbb{D}_E^* \cong \mathbb{D}_F^*$, and we claim that it takes radius q^{-r} to radius $q^{-1/r}$. Why is this? Suppose we have an analytic function $\sum a_m \pi_F^m$ on $\mathbb{D}_F^*$, with $a_m \in \mathbb{F}_q((\pi_E))$. We can then expand this as

$$\sum_{n\in\mathbb{Z}}\left(\sum_{m\in\mathbb{Z}} a_{n,m}\pi_E^m\right)\pi_F^n$$

with $a_{n,m} \in \mathbb{F}_q$. Can then switch the order of summation and get that this equals

$$\sum_{m\in\mathbb{Z}}\left(\sum_{n\in\mathbb{Z}} a_{n,m}\pi_F^n\right)\pi_E^m.$$

But doing this switches from Gauss norm $|\cdot|_{q^{-r}}$ to $|\cdot|_{q^{-1/r}}$.