Relative p-adic Hodge theory

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Hot Topics: Perfectoid Spaces and their Applications
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The term "p-adic Hodge theory" encompasses two aspects:

- *external* p-adic Hodge theory: comparison of cohomology theorems (étale, de Rham, crystalline, etc.) for algebraic varieties over p-adic fields; or
- *internal* p-adic Hodge theory: analysis of continuous p-adic representations of Galois groups of p-adic fields, including but not limited to étale cohomology of algebraic varieties.

In this talk, only the internal theory is considered. For the external theory in a similar relative setting, see Nizioł's lectures.

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Galois representations

Let K be a *p-adic field* (a field of characteristic 0 complete for a discrete valuation whose residue field is perfect of characteristic p) with absolute Galois group G_K.

In p-adic Hodge theory, one studies the categories $\mathbf{Rep}_{\mathbb{Z}_p}(G_K)$ and $\mathbf{Rep}_{\mathbb{Q}_p}(G_K)$ of continuous representations of G_K on finitely generated $\mathbb{Z}_p$-modules and $\mathbb{Q}_p$-modules. Note that the latter is the isogeny category of the former; that is, every $\mathbb{Q}_p$-representation admits G_K-stable lattices.

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Local systems on analytic spaces

Let X be an adic space locally of finite type over K. For $* \in \{\mathbb{Z}_p, \mathbb{Q}_p\}$, by an *étale $*$-local system* on X, we will mean a sheaf on $X_{\text{proét}}$ which is pro-étale locally of the form

$$\underline{V} : Y \mapsto \mathrm{Map}_{\mathrm{cont}}(|Y|, V) \qquad (\text{“constant sheaf”})$$

for V a finitely generated $*$-module with its usual topology. For instance, if $X = \mathrm{Spa}(K, K^\circ)$ this is just an object of $\mathbf{Rep}_*(G_K)$: take the neighborhood $Y = \mathrm{Spa}(\mathbb{C}_K, \mathbb{C}_K^\circ)$.

In general, étale $\mathbb{Q}_p$-local systems are not simply $\mathbb{Z}_p$-local systems up to isogeny! There are many natural examples arising from étale covers with noncompact groups of deck transformations: Tate uniformization of elliptic curves, Drinfel'd uniformizations, Rapaport-Zink period morphisms...

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Local systems via perfectoid spaces

For K a p-adic field, one studies $\mathbf{Rep}_{\mathbb{Q}_p}(G_K)$ by passing from K to some sufficiently ramified (*strictly arithmetically profinite*) algebraic extension K_∞ of K. Then $\widehat{K_\infty}$ is perfectoid; by tilting (and Krasner's lemma)

$$\mathbf{Rep}_{\mathbb{Q}_p}(G_{K_\infty}) \cong \mathbf{Rep}_{\mathbb{Q}_p}(G_{\widehat{K_\infty}}) \cong \mathbf{Rep}_{\mathbb{Q}_p}(G_{\widehat{K_\infty}^\flat})$$

so we can use the Frobenius on $\widehat{K_\infty}^\flat$ to study this category.

To study $\mathbf{Rep}_{\mathbb{Q}_p}(G_K)$, one must add descent data; often one takes K_∞/K Galois with $\Gamma = \mathrm{Gal}(K_\infty/K)$ a p-adic Lie group (e.g., $K_\infty = K(\mu_{p^\infty})$ with $\Gamma \subseteq \mathbb{Z}_p^\times$), and the descent data becomes a Γ-action.

But descent data can also be viewed as a sheaf condition for the pro-étale topology, in which case we can consider all choices for K_∞ at once! This point of view adapts well to analytic spaces, using perfectoid algebras as the analogue of strictly APF extensions.

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A simplifying assumption

Hereafter, X is an adic space over $\mathbb{Q}_p$ which is *uniform*: it is locally $\mathrm{Spa}(A, A^+)$ where A is a Banach algebra over $\mathbb{Q}_p$ whose norm is *submultiplicative* ($|xy| \leq |x||y|$) and *power-multiplicative* ($|x^2| = |x|^2$). In particular X is reduced. This restriction is harmless for our purposes:

- Any perfectoid space is uniform.
- For any adic space X over $\mathbb{Q}_p$, there is a unique closed immersed subspace X^u of X which is uniform and satisfies $|X^u| = |X|$, $X^u_{\text{ét}} \cong X_{\text{ét}}$, and $X^u_{\text{proét}} \cong X_{\text{proét}}$.
- Any adic space coming from a reduced rigid analytic space or a reduced Berkovich strictly[1] analytic space has this property.

Our constructions generally do not see A^+; this is related to the fact that $\mathrm{Spa}(A, A^\circ) \to \mathrm{Spa}(A, A^+)$ retracts onto its subspace of rank 1 valuations.

[1] Berkovich's non-strictly analytic spaces do not correspond to adic spaces; one needs a parallel adic theory where elements of value groups are always comparable with $\mathbb{R}$.

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Affinoid perfectoid subspaces

For this section, let's assume[2] that X is locally (strongly) noetherian. Then we may associate to X its *pro-étale topology* $X_{\text{proét}}$ as in de Jong's lecture.

For $Y = (Y_i) \in X_{\text{proét}}$, the *structure sheaf* on $X_{\text{proét}}$ is

$$\mathcal{O}_X : Y \mapsto \varinjlim_i \mathcal{O}(Y_i).$$

Each term in this limit inherits a power-multiplicative norm, its *spectral norm*. This norm is also the supremum over the valuations in Y_i, normalized p-adically.

Recall from de Jong's lecture that $X_{\text{proét}}$ has a neighborhood basis consisting of *affinoid perfectoid* subspaces (i.e., each Y_i comes from an adic ring and the completed inverse limit of these is perfectoid).

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The completed structure sheaf

From now on, let Y denote an arbitrary affinoid perfectoid in $X_{\text{proét}}$. We will specify a number of additional sheaves on $X_{\text{proét}}$ in terms of their values on Y; no promises are made about values on other pro-étale opens.

Proposition-Definition

There is a sheaf $\widehat{\mathcal{O}}_X$ on $X_{\text{proét}}$ such that $\widehat{\mathcal{O}}_X(Y)$ is the completion of $\mathcal{O}(Y)$ for the spectral norm.

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There is a sheaf $\overline{\mathcal{O}}_X$ on $X_{\text{proét}}$ such that $\overline{\mathcal{O}}_X(Y) = \widehat{\mathcal{O}}(Y)^\flat$.

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Sheaves of (overconvergent) Witt vectors

For R a perfect ring of characteristic p, the ring $W(R)$ of Witt vectors is p-adically separated and complete and $W(R)/(p) = R$. Reduction modulo p admits a multiplicative section, the Teichmüller map $\overline{x} \mapsto [\overline{x}]$.

Proposition-Definition

There is a sheaf $\tilde{\mathbf{A}}_X$ on $X_{\text{proét}}$ such that $\tilde{\mathbf{A}}_X(Y) = W(\overline{\mathcal{O}}_X(Y))$.

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If R carries a power-multiplicative norm, then for $r > 0$, the set $W^r(R)$ of $x = \sum_{n=0}^{\infty} p^n [\overline{x}_n] \in W(R)$ with $\lim_{n \to \infty} p^n |\overline{x}_n|^r = 0$ is a subring of $W(R)$.

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For any $r > 0$, there is a sheaf $\tilde{\mathbf{A}}_X^{\dagger,r}$ on $X_{\text{proét}}$ such that $\tilde{\mathbf{A}}_X^{\dagger,r}(Y) = W^r(\overline{\mathcal{O}}_X(Y))$. Put $\tilde{\mathbf{A}}_X^{\dagger} = \varinjlim_{r \to 0^+} \tilde{\mathbf{A}}_X^{\dagger,r}$.

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Nonabelian Artin-Schreier theory

For S a ring and φ an automorphism, a *φ-module* over S is a finite projective S-module M equipped with an isomorphism $\varphi^* M \cong M$ (i.e., a bijective semilinear φ-action).

Theorem (after Katz, SGA 7)

Let R be a perfect $\mathbb{F}_p$-algebra. The following categories are equivalent:

- *étale $\mathbb{Z}_p$-local systems on $\operatorname{Spec}(R)$;*
- *φ-modules over $W(R)$;*
- *φ-modules over $W^\dagger(R) = \cup_{r>0} W^r(R)$.*

For $R = F$ a field, the functors between étale $\mathbb{Z}_p$-local systems (identified with $\mathbf{Rep}_{\mathbb{Z}_p}(G_F)$) and φ-modules over $W(F)$ are

$$V \mapsto (V \otimes_{\mathbb{Z}_p} W(\overline{F}))^{G_F}, \qquad M \mapsto (M \otimes_{W(F)} W(\overline{F}))^{\varphi=1}$$

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For $R = F$ a field, the functors between étale $\mathbb{Z}_p$-local systems (identified with $\mathbf{Rep}_{\mathbb{Z}_p}(G_F)$) and φ-modules over $W(F)$ are

$$V \mapsto (V \otimes_{\mathbb{Z}_p} W(\overline{F}))^{G_F}, \qquad M \mapsto (M \otimes_{W(F)} W(\overline{F}))^{\varphi=1}$$

and similarly for $W^\dagger(F)$.

Nonabelian Artin-Schreier theory

For S a ring and φ an automorphism, a *φ-module* over S is a finite projective S-module M equipped with an isomorphism $\varphi^* M \cong M$ (i.e., a bijective semilinear φ-action).

Theorem (after Katz, SGA 7)

Let R be a perfect $\mathbb{F}_p$-algebra. The following categories are equivalent:

- *étale $\mathbb{Z}_p$-local systems on* $\operatorname{Spec}(R)$*;*
- *φ-modules over $W(R)$;*
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Sheafified φ-modules

A *φ-module* over a ring sheaf $*_X$ on $X_{\text{proét}}$ is a "quasicoherent finite projective"[3] sheaf $\mathcal{F}$ of $*_X$-modules plus an isomorphism $\varphi^*\mathcal{F} \cong \mathcal{F}$.

Proposition

Quasicoherent finite projective modules over $\tilde{\mathbf{A}}_X\Big|_Y$ or $\tilde{\mathbf{A}}_X^\dagger\Big|_Y$ correspond to finite projective modules over $\tilde{\mathbf{A}}_X(Y)$ or $\tilde{\mathbf{A}}_X^\dagger(Y)$, respectively. Moreover, these sheaves are acyclic.

[3] I.e., locally arises from a finite projective $*_X$-module. Because our rings are highly nonnoetherian, rational localizations may not be flat and so coherent sheaves cannot be handled easily, but vector bundles are no problem.

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$$T \mapsto T \otimes_{\underline{\mathbb{Z}_p}} \tilde{\mathbf{A}}_X, \qquad M \mapsto M^{\varphi=1}$$

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The analogue of taking Galois invariants in the first functor is the fact that the restriction of a φ-module to Y is defined by a finite projective module.

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Cohomology of $\mathbb{Z}_p$-local systems

By the *étale cohomology* of a local system, we will mean the ordinary cohomology on $X_{\text{proét}}$.

Theorem

For T an étale $\mathbb{Z}_p$-local system on X corresponding to a φ-module $\mathcal{F}$ over $\tilde{\mathbf{A}}_X$ and a φ-module $\mathcal{F}^\dagger$ over $\tilde{\mathbf{A}}_X^\dagger$, the sequences

$$0 \to T \to \mathcal{F}^* \stackrel{\varphi-1}{\to} \mathcal{F}^* \to 0 \qquad (* \in \{\emptyset, \dagger\})$$

are exact.

The point is that $\mathcal{F}^*$ is acyclic on *every* affinoid perfectoid, not just sufficiently small ones. (This recovers Herr's formula for Galois cohomology of $\mathbb{Z}_p$-local systems over a p-adic field.)

Contents

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Some more period sheaves

Following the previous analogy, we now define some more sheaves.

Proposition-Definition

There exist sheaves on $X_{\text{proét}}$ with the following sections.

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A φ-module over $\tilde{\mathbf{C}}_X$ is *étale* at $x \in X$ if adic-locally around x it arises by base extension from a φ-module over $\tilde{\mathbf{A}}^{\dagger}_X$.

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The étale condition is **pointwise***: it suffices to check it after pullback to the one-point space x.*

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The **slope polygon** *(to be defined later) of any φ-module is a lower semicontinuous function on X (with locally constant endpoints). If X arose from a Berkovich space, this is also true for Berkovich's topology (i.e., on the maximal Hausdorff quotient of X).*

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Globalized Artin-Schreier

Theorem

The following categories are equivalent:

- *étale $\mathbb{Q}_p$-local systems on X;*
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Also, for V an étale $\mathbb{Q}_p$-local system on X corresponding to a φ-module $\mathcal{F}$ over $\tilde{\mathbf{C}}_X^*$, for $* \in \{\emptyset, \infty\}$, the sequence

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Future attractions: removing the puncture

One can also define sheaves $\tilde{\mathbf{A}}^+_X, \tilde{\mathbf{B}}^+_X, \tilde{\mathbf{C}}^+_X$ where $\tilde{\mathbf{A}}^+_X(Y) = W(\overline{\mathcal{O}}(Y)^+)$. This is analogous to taking the whole unit disc, without a puncture.

One can define "Wach-Breuil-Kisin modules" over $\tilde{\mathbf{A}}^+_X$ where the action of φ is not bijective, but has controlled kernel and cokernel. These give rise to what we should call *crystalline* φ-modules over $\tilde{\mathbf{C}}_X$.

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The construction over an affinoid perfectoid

Pick any $r > 0$. The *relative Fargues-Fontaine curve* FF_Y is obtained[4] from the "annulus" $\mathrm{Spa}(\tilde{\mathbf{C}}_X^{[r/p,r]}(Y))$ by glueing the "edges" $\mathrm{Spa}(\tilde{\mathbf{C}}_X^{[r/p,r/p]}(Y))$ and $\mathrm{Spa}(\tilde{\mathbf{C}}_X^{[r,r]}(Y))$ via φ. This is independent of r. There is also an algebraic analogue:

$$\mathrm{FF}_Y^{\mathrm{alg}} = \mathrm{Proj}(P_Y), \qquad P_Y = \bigoplus_{n=0}^{\infty} \tilde{\mathbf{C}}_X(Y)^{\varphi = p^n}.$$

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There is a natural morphism $\mathrm{FF}_Y \to \mathrm{FF}_Y^{\mathrm{alg}}$ of locally ringed spaces which induces an equivalence of categories of vector bundles. Moreover, these categories are equivalent to φ-modules over $\tilde{\mathbf{C}}_Y$ and $\tilde{\mathbf{C}}_Y^{\infty}$. (Again, we don't consider coherent sheaves due to non-noetherianity.)

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Slopes over a perfectoid field

Suppose $X = \mathrm{Spa}(K, K^+)$ for K a perfectoid field; then FF_X is the Fargues-Fontaine *adic curve* associated to $K^\flat$. The algebraic curve $\mathrm{FF}_X^{\mathrm{alg}}$ is a noetherian scheme of dimension 1 with a morphism $\deg : \mathrm{Pic}(\mathrm{FF}_X) = \mathrm{Pic}(\mathrm{FF}_X^{\mathrm{alg}}) \to \mathbb{Z}$ taking $\mathcal{O}(1)$ to 1.

For any nonzero vector bundle $\mathcal{F}$ on FF_X, set

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The *slope* of $\mathcal{F}$ is $\mu(\mathcal{F}) = \deg(\mathcal{F})/\mathrm{rank}(\mathcal{F})$.

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A family of curves, in a sense

For general X, we may glue the adic (but not the algebraic) construction.

Theorem

For X perfectoid, the spaces FF_Y glue to give an adic space FF_X over $\mathbb{Q}_p$ which is **preperfectoid** *(its base extension from $\mathbb{Q}_p$ to any perfectoid field is perfectoid). The vector bundles on FF_X correspond to φ-modules over $\tilde{\mathbf{C}}_X$. Everything is functorial in X (and so far even in $X^\flat$).*

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Combining previous statements, we get the following.

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Ampleness for vector bundles

A vector bundle $\mathcal{F}$ on FF_X is **ample** if for any vector bundle $\mathcal{G}$ on FF_Y, $\mathcal{G} \otimes \mathcal{F}^{\otimes n}$ is generated by global sections for $n \gg 0$.

Theorem

$\mathcal{O}(1)$ is ample. Consequently, to check ampleness we need only consider $\mathcal{G} = \mathcal{O}(d)$ for $d \in \mathbb{Z}$ (over all Y; the powers of $\mathcal{F}$ need not be uniform).

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A distinguished section

So far, FF_X has been defined entirely in terms of $X^\flat$ (as in the lecture of Fargues). But it does admit some structures that depend on X:

- a distinguished ample line bundle $\mathcal{L}_X$ of rank 1 and degree 1;
- a distinguished section t_X of $\mathcal{L}_X$.

The zero locus of t_X is the image of a section $X \to \mathrm{FF}_X$ of the map $|\mathrm{FF}_X| \to |X|$. Unlike the fiber map, though, this is a map of adic spaces.

It should be possible to define sheaves $\mathbf{B}_{\mathrm{dR}}, \mathbf{B}_{\mathrm{crys}}, \mathbf{B}_{\mathrm{st}}$; for instance,

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Example: for $K = \mathbb{Q}_p$, $K_\infty = \mathbb{Q}_p(\mu_{p^\infty})$, we get $L = \mathbb{F}_p((\overline{\pi}))$.

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The example of (φ, Γ)-modules

For $K = \mathbb{Q}_p$, $K_\infty = \mathbb{Q}_p(\mu_{p^\infty})$, map $\mathbf{A} = \varprojlim_{n \to \infty} (\mathbb{Z}/p^n\mathbb{Z})((\pi))$ into $\tilde{\mathbf{A}} = \tilde{\mathbf{A}}_{\widehat{K_\infty}}$ by taking $1 + \pi$ to $[1 + \overline{\pi}]$. Then Γ lifts to $\mathbf{A}$ and $\mathbf{A}^\dagger$.

Theorem (Cherbonnier-Colmez)

The categories of (φ, Γ)-modules (φ-modules with compatible Γ-action) over $\mathbf{A}, \mathbf{A}^\dagger, \tilde{\mathbf{A}}, \tilde{\mathbf{A}}^\dagger$ are all equivalent.

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One may hope for an analogue of Cherbonnier-Colmez for other perfectoid towers, i.e., descent of φ-modules with descent data from $\tilde{\mathbf{A}}$ to some appropriate imperfect subring $\mathbf{A}$. This would perhaps give additional locally analytic representations sought by Berger-Colmez. (One may also want to descent from $\tilde{\mathbf{C}}$ to a suitable $\mathbf{C}$.)

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