MSRI Hot Topics Workshop: Perfectoid Spaces and their Applications

p-adic Hodge Theory for Rigid Spaces I - Wiesława Nizioł

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Summary: The aim of this lecture is to describe the proof of an important theorem about finiteness of étale cohomology modulo p for adic spaces over certain perfectoid fields, and a comparison theorem for this cohomology. The main technique here is to use a version of the Artin-Schreier sequence that is suitable for the sheaves we are dealing with. This reduces to needing to prove that a certain étale cohomology group is almost finitely generated, which is done by an analysis involving group cohomology and some spectral sequences.

Theorem 1. *Let $C/\mathbb{Q}_p$ be an algebraically closed complete extension, $X \to \mathrm{Spa}(C, \mathcal{O}_C)$ a proper smooth adic space, and let $\mathcal{L}$ be a $\mathbb{F}_p$-local system on X. Then:*

1. *$H^i(X_{\text{ét}}, \mathcal{L})$ is a finite-dimensional $\mathbb{F}_p$-vector space for all $i \geq 0$ and moreover vanishes for $i > d = 2\dim X$.*
2. *There is an almost isomorphism of $\mathcal{O}_C$-modules between $H^*(X_{\text{ét}}, \mathcal{L}) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p$ and $H^*(X, \mathcal{L} \otimes \mathcal{O}_X^+/p)$.*

This will be the main theorem that the talk is aimed at proving. Remarks:
(a) There's a relative version that follows from a base change theorem and (more general version of) the main theorem.
(b) The main theorem implies a B_{dR}-comparison theorem.

Proof: We will proceed by proving the almost isomorphism and finite-dimensionality together, then get vanishing. We will reduce to coherent cohomology via Artin-Schreier sequences, applying tilting and the pro-étale site. We recall what properties of the pro-étale site we'll use:

(1) $X_{\text{proét}}$ computes étale cohomology; if $\nu : X_{\text{proét}} \to X_{\text{ét}}$ is the natural map and $\mathcal{F}$ is a sheaf on $X_{\text{ét}}$, then $\mathcal{F} \cong R\nu_*\nu^*\mathcal{F}$.
(2) Over a perfectoid (K, K^+), affinoid perfectoids form a basis for the pro-étale topology.

(3) If U is an affinoid perfectoid then $H^i(U, \mathcal{O}_X^+/p)$ is almost isomorphic to 0 for $i > 0$. (This can be reduced to proving $H^i(W_{\text{ét}}, \mathcal{O}_X^+/p)$ is almost zero for W an affinoid perfectoid space, and this reduces to showing $H^i(W, \mathcal{O}_X^+/p)$ is almost zero. Note that an "affinoid perfectoid" is something in the pro-étale site and an "affinoid perfectoid space" is actually a space!).
(3') A twisted version of (3): If $\mathcal{L}$ is a $\mathbb{F}_p$-local system then $H^i(U, \mathcal{L} \otimes \mathcal{O}_X^+/p)$ is almost zero for $i > 0$. (When we pass from our pro-étale object U to an actual adic space $\widehat{U} = \mathrm{Spa}(S, S^+)$, we have $H^0(U, \mathcal{L} \otimes \mathcal{O}_X^+/p) = M(U)$ for $M(U)$ an almost finitely generated projective S^+/p-module, and compatibility with base change).

Want to use the *classical* Artin-Schreier sequence on $X_{\text{proét}}$,

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{O}_X^+/p \xrightarrow{\varphi - 1} \mathcal{L} \otimes \mathcal{O}_X^+/p \longrightarrow 0$$

This is exact; follows from looking locally; $\mathcal{L}_U$ is trivial for $U \in X_{\text{proét}}$ affinoid perfectoid, and by tilting it's enough to show surjectivity, which we'll do by reduction to characteristic p. So take U, the associated space $\widehat{U}$, and its tilt $\widehat{U}^\flat$, and pick $\pi \in \mathcal{O}^+_{\widehat{U}^\flat}$ with $\pi^\sharp = p$. Now have $\varphi - 1 : \mathcal{O}_U^+/p \to \mathcal{O}_U^+/p$ étale, and tilt this to $\mathcal{O}^+_{U^\flat}/p \to \mathcal{O}^+_{U^\flat}/p$. Pass from the tilted site back by using $\widehat{U}^\flat_{\text{ét}} \cong \widehat{U}^\flat_{\text{ét}}$. Get:

$$\cdots \longrightarrow H^i(X_{\text{ét}}, \mathcal{L}) \longrightarrow H^i(X_{\text{ét}}, \mathcal{L} \otimes \mathcal{O}_X^+/p) \xrightarrow{\varphi-1} H^i(X_{\text{ét}}, \mathcal{L} \otimes \mathcal{O}_X^+/p) \longrightarrow \cdots$$

Now, $H^i(X_{\text{ét}}, \mathcal{L} \otimes \mathcal{O}_X^+/p)$ is almost isomorphic to $(\mathcal{O}_C/p)^r$ for some r; this follows from the claim (to be proven later) that $H^i(X_{\text{ét}}, \mathcal{L} \otimes \mathcal{O}_X^+/p)$ is an almost-finitely generated $\mathcal{O}_C/p$-module, and using the Frobenius. So, we get a diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^i(X_{\text{ét}}, \mathcal{L}) & \longrightarrow & H^i(X_{\text{ét}}, \mathcal{L} \otimes \mathcal{O}_X^+/p) & \xrightarrow{\varphi-1} & H^i(X_{\text{ét}}, \mathcal{L} \otimes \mathcal{O}_X^+/p) & \longrightarrow \cdots \\
& \vdots & & \downarrow \text{almost-}\cong & & \downarrow \text{almost-}\cong & \\
0 \longrightarrow & \mathbb{F}_p^r & \longrightarrow & (C^\flat)^r & \xrightarrow{\varphi-1} & (C^\flat)^r & \longrightarrow 0
\end{array}$$

If we have vertical almost isomorphisms, and if there's a finiteness or rigidity statement we can get a map $H^i(X_{\text{ét}}, \mathcal{L}) \cong \mathbb{F}_p^r$ that's an actual isomorphism. So, we want to modify our setup and use a slightly different Artin-Schreier sequence (which will kill the "almost" in the isomorphisms above). We take the sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L} \otimes \widehat{\mathcal{O}}_{X^\flat} \xrightarrow{\varphi-1} \mathcal{L} \otimes \widehat{\mathcal{O}}_{X^\flat} \longrightarrow 0$$

Here $\widehat{\mathcal{O}}^+_{X^\flat}$ is the tilt $\varprojlim \mathcal{O}_X^+/p$ and $\widehat{\mathcal{O}}_{X^\flat} = \widehat{\mathcal{O}}^+_{X^\flat} \otimes_{\mathcal{O}_{C^\flat}} C^\flat$. Get a long exact sequence

$$\cdots \longrightarrow H^i(X_{\text{ét}}, \mathcal{L}) \longrightarrow H^i(X_{\text{ét}}, \mathcal{L} \otimes \widehat{\mathcal{O}}_{X^\flat}) \xrightarrow{\varphi-1} H^i(X_{\text{ét}}, \mathcal{L} \otimes \widehat{\mathcal{O}}_{X^\flat}) \longrightarrow \cdots$$

which has (actual, not almost) vertical isomorphisms to

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^i(X_{\text{ét}},\mathcal{L}) & \longrightarrow & H^i(X_{\text{ét}},\mathcal{L}\otimes\widehat{\mathcal{O}}_{X^\flat}) & \xrightarrow{\varphi-1} & H^i(X_{\text{ét}},\mathcal{L}\otimes\widehat{\mathcal{O}}_{X^\flat}) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{F}_p^r & \longrightarrow & (C^\flat)^r & \xrightarrow{\varphi-1} & (C^\flat)^r & \longrightarrow & 0 \end{array}$$

This proves the finite-dimensionality statement of part (1) of the theorem as well as part (2) of the theorem, modulo the claim that $H^i(X_{\text{ét}},\mathcal{L}\otimes\mathcal{O}_X^+/p)$ is almost-finitely-generated. We now want to prove the vanishing part of (1). Suffices to show that $H^i(X_{\text{ét}},\mathcal{L}\otimes\mathcal{O}_X^+/p)$ vanishes for $j>d$. Look at $\varepsilon: X_{\text{ét}}\to X_{\text{an}}$. Since the dimension of X_{an} is d it suffices to show that

$$R^j\varepsilon_*(\mathcal{L}\otimes\mathcal{O}_X^+/p)$$

is almost zero for $j>d$. Can make this computation locally, so can assume without loss of generality that X is affinoid, smooth, and has "good coordinates" (i.e. a map $X\to\mathbb{T}^d$ which is a composition of rational embeddings and finite étale maps). Now, $\mathbb{T}=\mathbb{T}^d$ is $\mathrm{Spa}(\mathbb{C}\langle T_i^{\pm1}\rangle,\mathcal{O}_C\langle T_i^{\pm1}\rangle)$. Take the perfection

$$\widetilde{\mathbb{T}}=\varprojlim\mathbb{T}=\mathrm{Spa}(\mathbb{C}\langle T_i^{\pm1/p^\infty}\rangle,\mathcal{O}_C\langle T_i^{\pm1/p^\infty}\rangle).$$

Take $\widetilde{X}=X\times_{\mathbb{T}}\widetilde{\mathbb{T}}$, which gives a $\mathbb{Z}_p^d$-pro-covering of X. Now,

$$R\Gamma(X_{\text{proét}},\mathcal{L}\otimes\mathcal{O}_X^+/p)=\mathcal{C}(\widetilde{X}/X,R\Gamma(\widetilde{X}^{\cdot}_{\text{proét}},\mathcal{L}\otimes\mathcal{O}_X^+/p)).$$

But

$$\begin{aligned} H^j(\widetilde{X}^{i/X}_{\text{proét}},\mathcal{L}\otimes\mathcal{O}_X^+/p) &= H^j(\widetilde{X}\times\mathbb{Z}_p^{d(i-1)},\mathcal{L}\otimes\mathcal{O}_X^+/p) \\ &= \mathrm{Hom}_{\text{cont}}(\mathbb{Z}_p^{d(i-1)},H^j(\widetilde{X},\mathcal{L}\otimes\mathcal{O}_X^+/p)), \end{aligned}$$

where we give the last cohomology group the discrete topology. This is almost zero for $j>0$, so we conclude that $H^j(X_{\text{proét}},\mathcal{L}\otimes\mathcal{O}_X^+/p)$ is almost isomorphic to $H^j_{\text{cont}}(\mathbb{Z}_p^d,M)$ where M is an almost-finitely-generated S^+-module. Since $\mathbb{Z}_p^d$ has cohomological dimension d, this continuous cohomology group vanishes for $j>d$, we get the desired result.

So it remains to prove the aforementioned claim. So let X be a proper smooth adic space over $\mathrm{Spa}(K,\mathcal{O}_K)$ where K is characteristic zero and contains all p-power roots of unity; want to prove $H^i(X_{\text{ét}},\mathcal{L}\otimes\mathcal{O}_X^+/p)$ is almost-finitely-generated as a $\mathcal{O}_K$-module. Idea: mimic the proof that coherent cohomology is finitely-generated for rigid analytic spaces in this setup.

Take a "good cover" $X=\bigcup V_i$, where each V_i is affinoid with good coordinates. By properness we can find another covering $\bigcup V_i'$ with $\overline{V}_i'\subseteq V_i$. Then have spectral sequences

$$E_1^{m_1,m_2}(V)=\bigoplus_{|J|=m_1+1}H^{m_2}(V_{J,\text{ét}},\mathcal{L}\otimes\mathcal{O}_X^+/p)\implies H^{m_1+m_2}(X,\mathcal{L}\otimes\mathcal{O}_X^+/p)$$

and

$$E_1^{m_1,m_2}(V') = \bigoplus_{|J|=m_1+1} H^{m_2}(V'_{J,\text{ét}}, \mathcal{L}\otimes\mathcal{O}_X^+/p) \implies H^{m_1+m_2}(X, \mathcal{L}\otimes\mathcal{O}_X^+/p),$$

and have natural restriction maps from the first sequence to the second. So suffices to prove that the image of

$$\mathrm{Res}_{m_1,m_2} : E_1^{m_1,m_2}(V) \to E_1^{m_1,m_2}(V')$$

is almost finitely generated. This gets a bit ugly...

Lemma 2. *Let V be good, V' a rational subset with $\overline{V}' \subseteq V$. Then the image of $\alpha : H^i(V_{\text{ét}}, \mathcal{L}\otimes\mathcal{O}_X^+/p) \to H^i(V'_{\text{ét}}, \mathcal{L}\otimes\mathcal{O}_X^+/p)$ is a finitely-generated $\mathcal{O}_K$-module.*

Proof. Taking our good coordinates and perfections we get a diagram

$$\begin{array}{ccccc} V' & \longrightarrow & V & \longrightarrow & \mathbb{T} \\ \uparrow & & \uparrow & & \uparrow \\ \widetilde{V}' & \longrightarrow & \widetilde{V} & \longrightarrow & \widetilde{\mathbb{T}}, \end{array}$$

with $\widehat{\widetilde{V}} = \mathrm{Spa}(S, S^+)$ and $\widehat{\widetilde{V}}' = \mathrm{Spa}(S', S'^+)$.

Step 1: Pass to group cohomology and get

$$\alpha : H^i_{\text{cont}}(\mathbb{Z}_p^d, M) \to H^i_{\text{cont}}(\mathbb{Z}_p^d, M \otimes S'^+/p)$$

where M is an almost finitely generated projective module over S^+/p. So can assume without loss of generality that $M = S^+/p$.

Step 2: Consider $(S_\mathfrak{m}, S_\mathfrak{m}^+)$ and $(S'_\mathfrak{m}, S'^+_\mathfrak{m})$. Suffices to show that for all $\mathfrak{m}$, the image of

$$H^i_{\text{cont}}(\mathbb{Z}_p^d, (S_\mathfrak{m}^+ \otimes R^+)/p) \to H^i_{\text{cont}}(\mathbb{Z}_p^d, (S'^+_\mathfrak{m} \otimes R^+)/p)$$

is almost finitely generated. Can do this by using the Hochschild-Serre spectral sequence and some more arguments... □