MSRI Hot Topics Workshop: Perfectoid Spaces and their Applications

Shimura Varieties and Perfectoid Spaces I: Completed Cohomology - Matthew Emerton

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Summary: This talk discusses some of the automorphic constructions that are important in Scholze's work on torsion classes. The main tool that is introduced is completed cohomology, which (when applied to Shimura varieties) will encode information about automorphic forms at all levels in a clean way. The idea is that, given a torsion class in the homology of a locally symmetric space, we can find the corresponding system of Hecke eigenvalues occurring in the completed cohomology for some Shimura variety, then use known theorems to conclude it comes from an automorphic representation. In particular, the speaker sketches how the classes we want can be found in the completed cohomology for a Shimura variety via studying the boundary of the Borel-Serre compactification.

In this talk, we will introduce some of the more automorphic aspects needed for Scholze's proof of the existence of Galois representations for torsion classes. Our setup will be that G is the $\mathbb{Z}_p$-points of some algebraic group, and G_r is the level p^r congruence subgroup. Then, we want a tower of manifolds

$$\to X_r \to X_{r-1} \to \cdots \to X$$

such that $X_r \to X$ is a torsor for G/G_r.

Examples:
(1) $G = \mathbb{Z}_P$, with a tower of S^1's with transition maps the p-power maps between them.
(2) $G = \mathrm{GL}_2(\mathbb{Z}_p)$, with tower having base the modular curve $Y(N)$ and r-th step is $Y(Np^r)$ (since $Y(Np^r) \to Y(N)$ is a $\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})$-torsor).

In this setup, we define the completed cohomology by

$$\widetilde{H}^i = \varprojlim_s \varinjlim_r H^i(X_r, \mathbb{Z}/p^s\mathbb{Z}).$$

This limit sees both Betti numbers and torsion in cohomology in an interesting way. Going back to the examples above:
(1) $\widetilde{H}^0 = \mathbb{Z}_p$ and $\widetilde{H}^1 = 0$ (because we're taking the inverse limit along the multiplication-by-p maps).
(2) $\widetilde{H}^0 = \mathbb{Z}_p[[(\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times]]$, and $\widetilde{H}^1$ is something interesting that encodes p-adic Langlands!

How do we relate the completed cohomology back to finite level? This is important e.g. in the modular curve case, where cohomology in finite level tells us about modular forms via Eichler-Shimura. We have a Hochschild-Serre spectral sequence $E_2^{ij} = H^i(G_r, \widetilde{H}^j) \implies H^{i+j}(X_r, \mathbb{Z}_p)$.

So suppose you had an algebra of operators (e.g. a Hecke algebra) on your cohomology and wanted to say something about the eigenvalues. If a system of eigenvalues appears at finite level, the spectral sequence says it should appear at infinite level; so it suffices to look there.

If W is a free $\mathbb{Z}_p$-module of finite rank with a continuous G-action, we get compatible local systems $\mathcal{W}_r/X_r$. Then we have that

$$E_2^{ij} = \mathrm{Ext}^i_{\mathbb{Z}_p[[G_r]]}(W^\vee, \widetilde{H}^j) \implies H^{i+j}(X_r, \mathcal{W}_r).$$

So even if we want to work with coordinates it's still sufficient to look just at completed cohomology.

For the context of Scholze's work, let G be a reductive group over $\mathbb{Q}$ (or over any number field, which we can restrict scalars down to $\mathbb{Q}$). Then, if K_f is the open compact in $G(\mathbb{A}_f)$ (for $\mathbb{A}_f$ the finite adeles), set

$$Y(K_f) = G(Q)\backslash G(\mathbb{A})/A_\infty^\circ K_\infty^\circ K_f,$$

where A_∞° is the connected component of $\mathbb{R}$-points of maximal $\mathbb{Q}$-split torus in the center of G and K_∞° the connected component of the maximal compact subgroup of $G(\mathbb{R})$.

Example: $G = \mathrm{GL}_2$. Then $A_\infty^\circ \cong \mathbb{R}_{>0}^\times$ (as scalar matrices) and $K_\infty^\circ = \mathrm{SO}_2(\mathbb{R})$. Then $\mathrm{GL}_2(\mathbb{R})/A_\infty^\circ K_\infty^\circ = \mathrm{GL}_2(\mathbb{R})/\mathbb{C}^\times$ is the usual representation of $\mathbb{C}\setminus\mathbb{R}$ as a quotient, and this is where we get a modular curve out of $Y(K_f)$.

Another example: if $G = \mathrm{GL}_2$ over an imaginary quadratic field, then $G(\mathbb{R}) = \mathrm{GL}_2(\mathbb{C})$, $A_\infty^\circ = \mathbb{R}_{>0}^\times$ and $K_\infty^\circ = U(2)$. Then we get

$$\mathrm{GL}_2(\mathbb{C})/A_\infty^\circ K_\infty^\circ = \mathrm{PSL}_2(\mathbb{C})/\mathrm{SO}(3) = \mathbb{H}^3,$$

hyperbolic 3-space. The quotients of these by congruence subgroups are Bianchi manifolds.

A theorem of Franke tells us that $H^i(Y(K_f), \mathbb{C})$ is computed by automorphic forms. This is a generalization of Eichler-Shimura, and a strengthening of Hodge theory to these non-compact manifolds. So the systems of Hecke eigenvalues that show up here are going to come from automorphic forms; and since we can also compute cohomology over $\mathbb{Z}$ they will be algebraic integers. So one might expect these to have Galois representations attached to them, and they do by work of Harris-Lan-Taylor-Thorne (most recently) and so on. But cohomology over $\mathbb{Z}$ might have torsion cohomology, and Scholze's work also assigns Galois representations to those. So in this way p-adic Hodge theory is really showing up in the same way that classical Hodge theory does!

Now, $\dim Y(K_f)$ is classically written as $2k_0 + \ell_0$ with

$$\ell_0 = \mathrm{rank}(G) - \mathrm{rank}(A_\infty^\circ) - \mathrm{rank}(K_\infty^\circ).$$

In the case where we have complex structure, ℓ_0 will be 0 and k_0 will be the complex dimension. This q_0 plays the role of the middle degree (which for algebraic varieties is the most important). So H^{q_0} is both the "first" and "last" interesting degree of cohomology; for lower degrees the systems of Hecke eigenvalues come from lower dimensional groups, and for higher degrees they show up from exterior powers of H^{q_0}. (This discussion only makes sense for cohomology with $\mathbb{C}$ coefficients).

Fix a ground level $K_f = K^p K_p$. Consider the tower of $Y(K^p K_p')$ (with $Y(K^p K_p)$ at the bottom) as K_p' varies over a sequence of compact open subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$ shrinking to 1. Can then form completed cohomology $\widetilde{H}^i$, and completed cohomology with compact supports $\widetilde{H}^i_c$. Then $G(\mathbb{Q}_p)$ acts on these, as does the Hecke algebra $\mathbb{T}$ generated by Hecke operators at primes $\ell \nmid pN$.

Conjecture 1 (Calegari-Emerton)**.** $\widetilde{H}^i = 0$ *for* $i > q_0$. *(Completed cohomology should strip away the "redundant information").*

One of Scholze's results shows that this is true for a wide class of Shimura varieties. How he proves this is by taking the tower $Y(K^p K_p')$ and realizing its inverse limit as a perfectoid space.

Now we talk a bit about compactifications and boundaries. Example: modular curve $Y_0(11)$ is topologically a torus with two points removed; compactify it by adding in cylinders (equivalently circles) at those points. So boundary becomes two circles. In general, the Borel-Serre compactification gives us a manifold with corners.

Example: Take $G = U(2,2)$; choose the quadratic imaginary field $F = \mathbb{Q}(i)$, look at $V = F^4$ with the Hermitian form $Q(x,y,z,w) = x\overline{y} - z\overline{w}$. Then $U(2,2)$ is the group of symmetries of this Q. This has a two-dimensional maximal torus, and two maximal parabolics: the Klingen parabolic (the stabilizer of an isotropy line) and a Siegel parabolic (the stabilizer of an isotropy plane). In the Klingen case, an isotropic line ℓ gives rise to a filtration $0 \subseteq \ell \subseteq \ell^\perp \subseteq V$, and on $\ell^\perp/\ell$ you get an induced form $\overline{Q}$ of type (1,1). Can check this has Levi isomorphic to $F^\times \times U(1,1)$. For the Siegel case, if W is an isotropic plane then $W = W^\perp$ and we get that the Levi is $\mathrm{GL}_2(F)$. Now, if we have our $Y_G = Y_G(K_f)$ sitting inside the Borel-Serre compactification $\overline{Y}_G$ with boundary ∂, then ∂ breaks up into two pieces, one for each parabolic P. Each of these pieces ∂_P is a nil bundle over

Y_M where M is a Levi of P. For $U(2,2)$ our Shimura variety is four-complex-dimensional so eight-real-dimensional, and the boundary is seven-dimensional. Each of the boundary pieces ∂_P is seven-dimensional (a certain bundle over the appropriate base space), and these are glued together along a six-dimensional nil bundle over circles. (A nil bundle is one coming from a successive extensions of $\mathbb{R}/\mathbb{Z}$).

Now look at cohomology; get a long exact sequence

$$\cdots \to H^i_c(Y_G) \to H^i(Y_G) \to H^i(\partial) \to H^{i+1}_c(Y_G) \to \cdots .$$

We have a map $H^i_c(\partial_P) \to H^i(\partial_P)$, and it factors through $H^i(\partial)$. From this long exact sequence, we see that to attach Galois representations to systems of Hecke eigenvalues appearing in $H^i(Y_M)$ (which appears in $H^i(\partial_P)$ from the nil bundles), we just need to do so for systems of Hecke eigenvalues appearing in $H^{\cdot}_c(Y_G)$. Since M is our Levi that gives us our symmetric spaces (e.g. Bianchi manifolds), that's the cohomology we really want to attach Galois representation to. By Hochschild-Serre it suffices to use systems of eigenvalues in $H^*_c(Y_G)$. But we can think of this as étale cohomology of a perfectoid space (a Shimura variety at "level p^∞"), and use a comparison theorem with coherent cohomology, and ultimately to classical modular forms on $U(2,2)$.