MSRI Hot Topics Workshop: Perfectoid Spaces and their Applications

# $p$-adic Hodge Theory for Rigid Spaces II - Wiesława Nizioł

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**Summary**: In this lecture the speaker introduces the Hodge-Tate spectral sequence for proper smooth rigid analytic varieties, and proves that it converges to de Rham cohomology. The argument proceeds in a number of steps; we first pass to the pro-étale topology, and then show that the resulting cohomology groups can be realized in terms of differentials (which ultimately relies on computations involving a complex of relative de Rham period rings, and a version of Faltings' extension). Finally, we discuss two natural short exact sequences that arise from this Hodge-Tate spectral sequence, which turn out to be dual to each other.

This talk will discuss the Hodge-Tate spectral sequence developed by Scholze. Let $C$ be a complete algebraically closed extension of $\mathbb{Q}_p$, and for most of the talk $X/C$ will be a proper smooth rigid analytic variety. Recall that we have the Hodge-de Rham spectral sequence

$$E_1^{ij} = H^j(X, \Omega_X^i) \implies H_{\text{dR}}^{i+j}(X),$$

coming from the natural Hodge filtration on the de Rham complex, $\operatorname{Fil}^k \Omega_X^{\cdot} = \Omega_X^{\geq k}$. If $X/C$ is a scheme, then the Hodge-de Rham spectral sequence degenerates, which can be proven in general by a "spreading out" argument to reduce to the case of a DVR. There's also another spectral sequence that we can form, the *Hodge-Tate spectral sequence.*

**Theorem 1.** *There is a Hodge-Tate spectral sequence*

$$E_2^{ij} = H^i(X, \Omega_X^j)(-j) \implies H_{\text{ét}}^{i+j}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C.$$

If $X/C$ is a scheme, then the Hodge-Tate spectral sequence degenerates at $E_2$; this should be true in general (without assuming it's a scheme). The sequence in the theorem is the descent spectral sequence for the projection $\nu : X_{\text{proét}} \to X_{\text{ét}}$.

*Step 1 of the Proof:*

$$H^i_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \cong H^i(X_{\text{proét}}, \widehat{\mathcal{O}}_X).$$

This follows from the basic comparison theorem proven yesterday, that we had an almost isomorphism

$$H^i_{\text{ét}}(X, \mathbb{Z}/p) \otimes_{\mathbb{Z}/p} \mathcal{O}_C/p \ \cong_a \ H^i(X_{\text{ét}}, \mathcal{O}_X^+/p).$$

We then use a devissage argument to get

$$H^i_{\text{ét}}(X, \mathbb{Z}/p^n) \otimes_{\mathbb{Z}/p^n} \mathcal{O}_C/p^n \ \cong_a \ H^i(X_{\text{ét}}, \mathcal{O}_X^+/p^n).$$

Then can take a direct limit, and get

$$H^i(X_{\text{proét}}, \widehat{\mathbb{Z}}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \ \cong_a \ H^i(X_{\text{proét}}, \mathcal{O}_X^+),$$

where $\widehat{\mathbb{Z}}_p = \varprojlim \mathbb{Z}/p^n$ on the pro-étale site. Finally, inverting $1/p$ gives the isomorphism we were looking for.

*Step 2:* Let $X/C$ be a smooth adic space. Then there exists a natural isomorphism $R^j\nu_*\widehat{\mathcal{O}}_X \cong \Omega^j_{X_{\text{ét}}}(-j)$. There are two steps to proving this, first getting the identification for $j = 1$ and then studying the exterior powers of that to get the identification for larger $j$.

We start by claiming that if $\mathcal{E} = R^1\nu_*\widehat{\mathcal{O}}_X$ is a locally free $\mathcal{O}_{X_{\text{ét}}}$-module of rank $d = \dim X$, such that $\bigwedge^j \mathcal{E} \cong R^j\nu_*\widehat{\mathcal{O}}_X$ for $j \geq 0$. We prove this by looking locally; assume $X \to \mathbb{T} = \mathbb{T}^d$ is a choice of good coordinates, where

$$\mathbb{T} = \text{Spa}(C\langle T_i^{\pm 1}\rangle, \mathcal{O}_C\langle T_i^{\pm 1}\rangle)$$

and this has a $\mathbb{Z}_p^d$-cover by

$$\widetilde{\mathbb{T}} = \text{Spa}(C\langle T_i^{\pm 1/p^\infty}\rangle, \mathcal{O}_C\langle T_i^{\pm 1/p^\infty}\rangle).$$

Then we have a $\mathbb{Z}_p^d$-cover $X \times_{\mathbb{T}} \widetilde{\mathbb{T}} = \widetilde{X} \to X$, and have

$$H^i(X_{\text{proét}}, \widehat{\mathcal{O}}_X) = H^i_{\text{cont}}(\mathbb{Z}_p^d, M)$$

where $M = \mathcal{O}_{\widetilde{X}}(\widetilde{X}) = \mathcal{O}_X(X) \otimes C\langle T_i^{\pm 1/p^\infty}\rangle$. Compute

$$H^i_{\text{cont}}(\mathbb{Z}_p^d, M) = \mathcal{O}_X(X) \widehat{\otimes} H^i_{\text{cont}}(\mathbb{Z}_p^d, C\langle T_i^{\pm 1/p^\infty}\rangle).$$

Next, we note that we have a map

$$\mathcal{O}_X(X) \widehat{\otimes} H^i_{\text{cont}}(\mathbb{Z}_p^d, C\langle T_i^{\pm 1}\rangle) \to \mathcal{O}_X(X) \widehat{\otimes} H^i_{\text{cont}}(\mathbb{Z}_p^d, C\langle T_i^{\pm 1/p^\infty}\rangle),$$

where $C\langle T_i^{\pm 1}\rangle$ has the trivial action. It turns out that this is an isomorphism, and then we have a further isomorphism.

$$\mathcal{O}_X(X) \widehat{\otimes} H^i_{\text{cont}}(\mathbb{Z}_p^d, C\langle T_i^{\pm 1}\rangle) \cong \mathcal{O}_X(X) \widehat{\otimes} \bigwedge^i (C\langle T_i^{\pm 1}\rangle)^d.$$

Hence we get $H^0(X_{\text{proét}}, \widehat{\mathcal{O}}_X) \cong \mathcal{O}_X(X)$, and

$$H^i(X_{\text{proét}}, \widehat{\mathcal{O}}_X) \cong \bigwedge^i H^1(X_{\text{proét}}, \widehat{\mathcal{O}}_X).$$

*Step 3:* We want to prove the identification $\mathcal{E} \cong \Omega^1_{X_{\text{ét}}}(-1)$. Why would we believe this is true? There's a Poincaré lemma in a simpler situation. Look at the case where $X$ is over $\mathrm{Spa}(K, \mathcal{O}_K)$ for $K$ a complete discretely valued field with perfect residue field. Then, we have an exact sequence of sheaves on the pro-étale site

$$0 \to \mathcal{B}^+_{\mathrm{dR}} \to \mathcal{O}\mathcal{B}^+_{\mathrm{dR}} \to \mathcal{O}\mathcal{B}^+_{\mathrm{dR}} \otimes \Omega^1_X \to \cdots .$$

Here, $\mathcal{B}^+_{\mathrm{dR}}$ is Fontaine's relative $B^+_{\mathrm{dR}}$. The ring $\mathcal{O}\mathcal{B}^+_{\mathrm{dR}}$ is locally $\mathcal{B}^+_{\mathrm{dR}}[[u_1, \ldots, u_d]]$ where $u_i = T_i \otimes 1 - 1 \otimes [T_i^\flat]$.

Remark: This is a rigid version of the Poincaré lemma that Faltings used to construct his period map. How did those arise? Suppose we had a smooth $X \to \mathrm{Spec}\, W(k)$, and a crystal $\mathcal{E}$ over $X$, which corresponds to a $\mathcal{F}$-vector bundle with a connection $\nabla$. Then there exists a complex

$$\mathcal{F} \to \mathcal{F} \otimes \Omega^1 \to \cdots .$$

Taking a "linearization of this complex" gives a resolution of $\mathcal{E}$ by acyclic crystals,

$$0 \to \mathcal{E} \to \mathcal{F}(\mathcal{F}) \to \mathcal{L}(\mathcal{F} \otimes \Omega^1) \to \cdots .$$

Evaluate this resolution on $A^+_{\mathrm{crys}}$ and get

$$0 \to \mathcal{E}(A^+_{\mathrm{crys}}) \to \mathcal{F}(\mathcal{F})(A^+_{\mathrm{crys}}) \to \mathcal{L}(\mathcal{F} \otimes \Omega^1)(A^+_{\mathrm{crys}}) \to \cdots .$$

Want to get Faltings' extension from this Poincaré lemma, looking at $\mathrm{gr}^1$. We can filter the Poincaré lemma by $(\ker \theta)^i$ and take $\mathrm{gr}^1$ to get

$$0 \to \widehat{\mathcal{O}}_X(1) \to \mathrm{gr}^1_F \mathcal{O}\mathcal{B}^+_{\mathrm{dR}} \to \widehat{\mathcal{O}}_X \otimes_{\mathcal{O}_X} \to 0$$

Once we have this, we apply the pushforward $R\nu_*$ to get the following portion of a long-exact sequence.

$$\nu_* \widehat{\mathcal{O}}_X(1) \to \nu_* \mathrm{gr}^1_F \mathcal{O}\mathcal{B}^+_{\mathrm{dR}} \to \nu_*(\widehat{\mathcal{O}}_X \otimes_{\mathcal{O}_X}) \to R^1\nu_* \widehat{\mathcal{O}}_X(1) \to R^1\nu_* \mathrm{gr}^1_F \mathcal{O}\mathcal{B}^+_{\mathrm{dR}}.$$

We want to show that the boundary map

$$\partial : \nu_*(\widehat{\mathcal{O}}_X \otimes_{\mathcal{O}_X}) \to R^1\nu_* \widehat{\mathcal{O}}_X(1)$$

is an isomorphism. This is equivalent to saying $\nu_* \mathrm{gr}^1_F \mathcal{O}\mathcal{B}^+_{\mathrm{dR}}$ and $R^1\nu_* \mathrm{gr}^1_F \mathcal{O}\mathcal{B}^+_{\mathrm{dR}}$ are zero. This is a computation, using the fact that

$$R^k\nu_* \mathrm{gr}^1_F \mathcal{O}\mathcal{B}^+_{\mathrm{dR}} = 0$$

for $k \geq 0$.

Example: $A/C$ an abelian variety. Then have Hodge-de Rham spectral sequence, which gives

$$0 \to H^0(A, \Omega^1) \to H^1_{\mathrm{dR}}(A) \to H^1(A, \mathcal{O}_A) \to 0.$$

Also have the Hodge-Tate exact sequence (which we call "HT1")

$$0 \to H^1(A, \mathcal{O}_A) \to H^1_{\text{ét}}(A, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C \to H^0(A, \Omega^1_A)(-1) \to 0$$

Want to describe this in the case where $A$ has good reduction, using the associated $p$-divisible group; so we have an abelian variety $A/\mathcal{O}_C$ and $G = A[p^\infty]$. We can write down another Hodge-Tate sequence, "HT2", from the following theorem.

**Theorem 2** (Faltings, Fargues)**.** *The complex of finite free $\mathcal{O}_C$-modules*

$$0 \to (\operatorname{Lie} G)(1) \to TG \otimes_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathcal{O}_C \to (\operatorname{Lie} G^*)^* \to 0$$

*has cohomology annihilated by $p^{1/(p-1)}$.*

Here $TG$ is the Tate module of $G$, given by $\varprojlim_n G[p^n](\mathbb{C})$. The map

$$\alpha_G : TG \otimes_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathcal{O}_C \to (\operatorname{Lie} G^*)^*$$

comes from taking $\alpha \in \varprojlim G[p^n](\mathcal{O}_C) \operatorname{Hom}_{\mathcal{O}_C}(\mathbb{Q}_p/\mathbb{Z}_p, G)$, dualizing to get $\alpha^* : G^* \to \mu_{p^\infty}$, and linearizing to get $\operatorname{Lie}(\alpha^*) : \operatorname{Lie}(G^*) \to \operatorname{Lie} \mu_{p^\infty} \cong \mathcal{O}_C$. This map has the rationality property that if $G$ is defined over a field $L$, then $\alpha_G$ can be defined over the field generated by $L$ and the torsion points.

**Theorem 3.** *The Hodge-Tate sequences HT1 and HT2 are dual.*

To prove this, re-write HT1 as

$$0 \to \operatorname{Lie} A^* \otimes_{\mathcal{O}_C} C \to H^1_{\text{ét}}(A_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C \to (\operatorname{Lie} A)^* \otimes_{\mathcal{O}_C} C(-1) \to 0.$$

We can then dualize, and write down a diagram of maps comparing to HT2 which can be verified to commute and have isomorphisms.