MSRI Hot Topics Workshop: Perfectoid Spaces and their Applications

Shimura Varieties and Perfectoid Spaces II - Mark Kisin

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Summary: In this lecture the speaker sketches the proof that one can realize an infinite limit of Shimura varieties as a perfectoid space, which is proved alongside showing the Hodge-Tate period map is a morphism of adic spaces. The proof begins by using a version of the theory of the canonical subgroup to construct an open set on which the infinite-level Shimura variety is perfectoid. We then extend this to showing that the whole space is perfectoid by using the natural group action on the Shimura variety.

The aim of these two talks is to show that the Hecke action on completed cohomology is controlled by the Hecke action on modular forms. That will be done in the second talk; this talk will focus on a critical tool, the Hodge-Tate period map. Fix a dimension $g \geq 1$ and fix a level away from $K^p \subseteq \mathrm{GSp}_{2g}(A_f^p)$ (which will be used implicitly throughout). Then have moduli spaces $X_{\Gamma(p^n)}$ and $X_{\Gamma_0(p^n)}$, parametrizing principally polarized abelian varieties of dimension g and the specified level, given their minimal compactifications. (In this talk, we'll often just do things away from the boundary; Scholze has to extend these constructions to the boundary in his paper). The main case to keep in mind for intuition is for elliptic curves, $g = 1$.

Continuing our setup, let (K, K^+) be a complete nonarchimedean field together with an open bounded subring. Want to consider $X_{\Gamma(p^\infty)}(K, K^+)$, the inverse limit $\varprojlim X_{\Gamma(p^n)}(K, K^+)$; eventually we'll make this a perfectoid space, but for now just look at the underlying set. A point $x \in X_{\Gamma(p^\infty)}(K, K^+)$ (not in the boundary) corresponds to the data of an abelian variety A/K along with an isomorphism $A[p^\infty](K) \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{2g}$. In particular can use the Hodge filtration on the Tate module plus this isomorphism to get a filtration

$$\mathrm{Lie}\, A(1) \subseteq T_p A \otimes K \cong \mathbb{Z}_p^{2g} \otimes K$$

defined over K. Thus we get an isotropic subspace of K^{2g}. If $\mathcal{F}$ is the Grassmannian of all isotropic g-dimensional subspaces of K^{2g}, we've determined a point of $\mathcal{F}(K)$. Thus we have a map

$$\pi_{HT} : X_{\Gamma(p^\infty)}(K) \to \mathcal{F}(K)$$

(which we've only defined only away from the boundary).

Theorem 1. *There is a perfectoid space $X_{\Gamma(p^\infty)}$ defined over $\mathbb{Q}_p^{\mathrm{cycl}}$ such that $X_{\Gamma(p^\infty)} \sim \varprojlim X_{\Gamma(p^n)}$, and π_{HT} defined above is induced from a $\mathrm{GSp}_{2g}(\mathbb{Q}_p)$-equivariant map of adic spaces $X_{\Gamma(p^\infty)} \to \mathcal{F}$. Moreover, if $(G, X) \subseteq (\mathrm{Gsp}, S^\pm)$ is a Shimura datum of Hodge type and we let X_G be the associated Shimura variety, can form a perfectoid space $X_{G,\Gamma(p^\infty)} \sim \varprojlim X_{G,\Gamma(p^n)}$ which has a period map to $\mathcal{F}$.*

Example of $g = 1$: the underlying set $|X_{\Gamma(p^\infty)}|$ breaks up as a disjoint union of the closure of the ordinary locus $\overline{X^{\mathrm{ord}}_{\Gamma(p^\infty)}}$ and the supersingular locus $X^{\mathrm{ss}}_{\Gamma(p^\infty)}$. Here $\mathcal{F}$ is just $\mathbb{P}^1$, and the period map sends the ordinary locus to $\mathbb{P}^1(\mathbb{Q}_p)$ and the supersingular locus to the complement, Drinfeld's upper half-space $\Omega = \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{Q}_p)$. Moreover, $X^{\mathrm{ss}}_{\Gamma(p^\infty)}$ is Lubin-Tate space at infinite level, and is isomorphic to Drinfeld space at infinite level; under this isomorphism, the period map just corresponds to the projection $\Omega_\infty \to \Omega$.

Main thing we want to explain is the first fact in the theorem, that $X_{\Gamma(p^\infty)}$ is a perfectoid space. That π_{HT} is a map of adic space is intertwined in this proof, but in a technical way. The idea of proving that this is a perfectoid space is proving perfectoidness on a certain open part of it and use the action of $\mathrm{GSp}_{2g}(\mathbb{Q}_p)$.

Fix some $0 \leq \varepsilon < 1$ and Γ to be either $\Gamma_0(p^\infty)$ or $\Gamma_1(p^\infty)$. Define subsets $X_\Gamma(0) \subseteq X_\Gamma(\varepsilon) \subseteq X_\Gamma$ where $X_\Gamma(\varepsilon)$ is the locus where the Hasse invariant H satisfies $|H| \geq p^{-\varepsilon}$. For the case $\Gamma = \Gamma(1)$ we'll just drop the level from the notation and write $X(0) \subseteq X(\varepsilon) \subseteq X$.

Proposition 2. *Let $0 \leq \varepsilon < 1/2$. Then:*

1. *For $m \geq 1$, the universal abelian variety $A(p^{-m}\varepsilon) \to X(p^{-m}\varepsilon)$ admits a "canonical subgroup" $C_m \subseteq A(p^{-m}\varepsilon)[p^m]$.*
2. *The map $A \mapsto A/C_1$ induces the map $\widetilde{F} : X(p^{-m}\varepsilon) \to X(p^{-m+1}\varepsilon)$ which reduces (along our natural integral models) to the Frobenius modulo $p^{1-\varepsilon}$.*
3. *If we map a pair (A, C_m) to $(A/C_m, A[p^m]/C_m)$ (viewing $A[p^m]/C_m$ as a $\Gamma_0(p^m)$-level structure), this induces a map $X(p^{-m}\varepsilon) \to X_{\Gamma_0(p^m)}$ that induces an isomorphism onto an open and closed subset $X_{\Gamma_0(p^m)}(\varepsilon)_a \subseteq X_{\Gamma_0(p^m)}(\varepsilon)$, where the a here stands for "anti-canonical".*

Combining the maps from the proposition, we can get a commutative diagram

$$\begin{array}{ccccc}
 & & X(p^{-m+1}\varepsilon) & \xrightarrow{A\mapsto A/C_{m+1}} & X_{\Gamma_0(p^{m+1})} \\
 & & \downarrow{\scriptstyle \widetilde{F}} & & \downarrow \\
 & & X(p^{-m}\varepsilon) & \xrightarrow{A\mapsto A/C_1} & X_{\Gamma_0(p^m)} \\
 & & \downarrow & & \downarrow \\
X(p^{-1}\varepsilon) & \xrightarrow{\cong} & X_{\Gamma_0(p)}(\varepsilon)_a & \longrightarrow & X_{\Gamma_0(p)}
\end{array}$$

We claim that this diagram is actually Cartesian. We're trying to show the tower on the right gives a perfectoid space; we'll show it by pulling back to $X(p^{-1}\varepsilon)$ and showing that the tower on the left-hand side is perfectoid. But that isn't too difficult from (2).

Definition 3. Define

$$X_{\Gamma_0(p^m)}(\varepsilon)_a = \varprojlim X_{\Gamma_0(p^m)}(\varepsilon)_a \cong \varprojlim_{\widetilde{F}} X(p^{-m}\varepsilon),$$

which is a perfectoid space by (2) of the proposition. Then, define a map $X_{\Gamma(p^\infty)} \to X_{\Gamma_0(p^\infty)}$ by mapping (A, α) (where $\alpha : T_pA \cong \mathbb{Z}_p^{2g}$ is an isomorphism) to $(A, \alpha^{-1}[\mathbb{Z}_p^g \oplus 0])$.

We define $X_{\Gamma(p^m)}(\varepsilon)_a$ as the preimage of $X_{\Gamma_0(p^m)}(\varepsilon)_a$ and $X_{\Gamma(p^\infty)}(\varepsilon)_a$ as the inverse limit of these $X_{\Gamma(p^m)}(\varepsilon)_a$; this is perfectoid. The locus where $X_{\Gamma(p^\infty)}$ is stable under $\mathrm{GSp}_{2g}(\mathbb{Q}_p)$, and $X_{\Gamma(p^\infty)}(\varepsilon)_a$ is perfectoid. Claim that if we let $\mathrm{Gsp}_{2g}(\mathbb{Z}_p)$ on this the orbit is $X_{\Gamma(p^\infty)}(\varepsilon)$ so we conclude that is perfectoid (removing the "anti-canonical").

Lemma 4. *The preimage $\pi_{HT}^{-1}[\mathcal{F}(\mathbb{Q}_p)]$ equals the closure $\overline{X_{\Gamma(p^\infty)}(0)}$. Moreover, there exists an open neighborhood U of $\mathcal{F}(\mathbb{Q}_p)$ in $\mathcal{F}$ such that $\pi_{HT}^{-1}(U) \subseteq X_{\Gamma(p^\infty)}(\varepsilon)$.*

Then, if we take U as in the lemma we find that $\mathrm{GSp}_{2g} \cdot X_{\Gamma(p^\infty)(\varepsilon)}$ contains $\pi_{HT}^{-1}(\mathrm{GSp}_{2g} \cdot U)$. So it's enough to show that $\mathrm{GSp}_{2g} \cdot U$ is all of $\mathcal{F}$.

Lemma 5. *If $U \subseteq \mathcal{F}$ is open and contains $F(\mathbb{Q}_p)$, and is stable under the action of $\mathrm{GSp}_{2g}(\mathbb{Q}_p)$, then $U = \mathcal{F}$.*

Proof for $g = 1$: Already assuming U contains the $\mathbb{Q}_p$-points, so can choose x corresponding to $\mathbb{Q}_p \oplus 0 \in \mathbb{P}^1(\mathbb{Q}_p)$. If B is a ball with $x \in B \subseteq \mathbb{P}^1 \setminus \{\infty\}$, then acting by the elements

$$\begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}^n$$

contracts the ball towards the point. Since U is open, it contains the image of B under one of these elements, and since U is invariant under $\mathrm{GSp}_{2g}(\mathbb{Q}_p)$ it contains B itself. This tells us $U = \mathcal{F}$.