MSRI Hot Topics Workshop: Perfectoid Spaces and their Applications

Shimura Varieties and Perfectoid Spaces III - Mark Kisin

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Notes taken by Dan Collins (djcollin@math.princeton.edu)

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Summary: This lecture discusses the proof that the action of an Hecke algebra on completed cohomology factors through the "classical" Hecke algebra coming from modular forms. This theorem is a key step in Scholze's work on torsion in homology, because it lets one pass between modular forms (which are realized in cohomology at finite level) to completed cohomology at infinite level (which can be studied via perfectoid spaces). The argument proceeds by passing from completed cohomology to cohomology of a perfectoid space, and then analyzing this by using the Hodge-Tate period map to construct a Čech cover by good affinoids on which we can work with the cohomology explicitly.

Start by setting up our notation: Let $(G, X) \subseteq (\mathrm{GSp}, S^{\pm})$ be a Shimura datum of Hodge type, and fix a level $K^p \subseteq G(\mathbb{A}_f^p)$ away from infinity and p. For a level $K_p = G(\mathbb{Q}_p)$ at p, we have the Shimura variety $Y_{K_pK^p}$ contained in its minimal compactification $X_{K_pK^p}$. Everything will be over $\mathbb{C}_p$.

Definition 1. Define some cohomology groups (with compact support, and fixed level away from p), as

$$\widetilde{H}^i_{c,K^p}(\mathbb{Z}/p^n\mathbb{Z}) = \varinjlim_{K_p} H^i_c(Y_{K_pK^p}, \mathbb{Z}/p^n\mathbb{Z}).$$

If we took the inverse limit over n we'd get the completed cohomology from Emerton's talk, but we won't do that. Also define a Hecke algebra

$$\mathbb{T} = \mathbb{Z}_p[K^p \backslash G(\mathbb{A}_f^p)/K^p],$$

which acts on $\widetilde{H}^i_{c,K^p}(\mathbb{Z}/p^n\mathbb{Z})$.

(We remark that $\widetilde{H}^i_{c,K^p}(\mathbb{Z}/p^n\mathbb{Z})$ will be equal to some cohomology group $H^i_{c,K^p}(Y, \mathbb{Z}/p^n\mathbb{Z})$ of a perfectoid space). On $X_{K_pK^p}$ we have a line bundle ω, coming from Ω^g from the universal family of abelian varieties. Also, let I be the ideal sheaf of $X \setminus Y$. The Hecke algebra $\mathbb{T}$ also acts on modular forms, for our purposes cusp forms in the space

$$H^0(X_{K_pK^p}, \omega^{\otimes k} \otimes I).$$

Theorem 2. *Let $\mathbb{T}_{\mathrm{cl}}$ be the maximal quotient of $\mathbb{T}$ such that the action of $\mathbb{T}$ on $H^0(X_{K_pK^p}, \omega^{\otimes k} \otimes I)$ factors through $\mathbb{T}_{\mathrm{cl}}$ for all k and all K_p. (Here we mean the ideal we divide by is maximal). Then the action of $\mathbb{T}$ on $\widetilde{H}^i_{c,K^p}(\mathbb{Z}/p^n\mathbb{Z})$ factors through $\mathbb{T}_{\mathrm{cl}}$.*

We now start to discuss the ingredients of the proof. Write

$$X = \varprojlim_{K_p} X_{K_pK^p},$$

which is what we denoted $X_{G,\Gamma(p^\infty)}$ in the last talk, which we know is a perfectoid space. We define $I^+ = \mathcal{O}^+ \cap I$. The following proposition is what is used to compute the completed cohomology:

Proposition 3. *There is an almost isomorphism*

$$\widetilde{H}^i_{c,K^p}(\mathbb{Z}/p^n\mathbb{Z}) \otimes \mathcal{O}_{\mathbb{C}_p}/p^n \;\cong_a\; H^i(X, I^+/p^nI^+).$$

If there's no boundary, then $I^+ = \mathcal{O}^+$, and the proposition was a result in some of the earlier talks. The case with boundary is a generalization of this; an idea of how to extend those results to this case comes from noting that I^+/p^nI^+ is the extension-by-zero $j_!\mathcal{O}^+/p\mathcal{O}^+$. Then could prove this by some sort of dévissage argument.

The upshot of the proposition is that, to prove the theorem, we can switch to computing with $H^i(X, I^+/p^nI^+)$; for this we'll use the Hodge-Tate period map and the fact X is perfectoid. Recall we had a variety $\mathcal{F}$ (parametrizing isotropic subspaces of dimension d) from the previous talk associated to the symplectic group; our period map $\pi_{HT} : X \to \mathcal{F}$ is the composition of the map $X \to X_{\mathrm{GSp}}$ with the period map on X_{GSp}. Also, $\mathcal{F}$ embeds in $\mathbb{P}^{\binom{2g}{g}-1}$ via the usual Plücker embedding; $D \subseteq \mathbb{Z}_p^{2g} \otimes K$ maps to $\bigwedge^g D \subseteq \bigwedge^g(K^{2g})$, with projective coordinates given by S_J for $J \subseteq \{1, \ldots, 2g\}$ with $|J| = g$. Define $\mathcal{F}_J \subseteq \mathcal{F}$ as the locus where $|S_J| \geq |S_{J'}|$ for all J'. We then note that $\mathcal{O}(1)$ on this projective space it pulls back to ω on X.

Example: for $J = \{g+1, \ldots, 2g\}$, we have $\pi_{HT}^{-1}[\mathcal{F}_J(\mathbb{Q}_p)] = \overline{X_{\Gamma(p^\infty)}(0)}_a$.

Lemma 4. *All of the $\pi_{HT}^{-1}[\mathcal{F}_J]$'s are affinoid perfectoid.*

Proof. Using the action of the group, it's enough to consider $J = \{g+1, \ldots, 2g\}$. Consider the diagonal matrix

$$\gamma = \begin{bmatrix} pI_g & 0 \\ 0 & I_g \end{bmatrix},$$

and consider $\gamma^n \mathcal{F}_J \subseteq \mathcal{F}_J$. Can check this is a rational subdomain (just a calculation on the Grassmannian). If $n \gg 0$, have

$$\pi_{HT}^{-1}[\gamma^n \mathcal{F}_J] \subseteq X_{\Gamma(p^\infty)}(\varepsilon)_a;$$

since $X_{\Gamma(p^\infty)}(\varepsilon)_a$ is affinoid we conclude that $\pi_{HT}^{-1}[\gamma^n \mathcal{F}_J]$ is affinoid. Since π_{HT} is GSp-equivariant, conclude $\pi_{HT}^{-1}[\mathcal{F}_J]$ is affinoid. □

By the lemma, can compute $H^i(X, I^+/p^n I^+)$ by using the open cover by affinoids $\{\pi_{HT}^{-1}[\mathcal{F}_J]\}_J$. We enumerate these affinoids as $\mathcal{V}_1, \dots, \mathcal{V}_N$ for $N = \binom{2g}{g}$, and for $J_2 \subseteq \{1, \dots, N\}$ set $\mathcal{V}_{J_2} = \bigcap_{j \in J_2} \mathcal{V}_j$. Then:

Theorem 5. *The cohomology group $H^i(\mathcal{V}_{J_2}, I^+/p^n I^+)$ is almost zero for $i > 0$.*

So it's enough to show that the action of $\mathbb{T}$ on $H^0(\mathcal{V}_{J_2}, I^+/p^n I^+)$ factors through $\mathbb{T}_{\mathrm{cl}}$ (since then we'd have that $\mathbb{T}$ factors through $\mathbb{T}_{\mathrm{cl}}$ for a Čech complex for the original cohomology groups we were trying to prove this for). Remark: The $\mathcal{V}_{J_2}$'s come from $\mathcal{V}_{J_2, K_p} \subseteq C_{K_p K^p}$ for K_p small enough.

So we have these $\mathcal{V}_{J_2, K_p}$'s which are affinoids. The bounded functions give a natural integral model $\mathcal{V}^\circ_{J_2, K_p}$. Can glue the $\mathcal{V}^\circ_{J_2, K_p}$'s to a formal scheme $X^\circ_{K_p K^p}$ equipped with an ample line bundle ω^{int} extending ω. Finally, need to compute

$$H^0(\mathcal{V}_{J_2}, I^+/p^n I^+) = \varinjlim_{K_p} H^0(\mathcal{V}_{J_2, K_p}, I^+/p^n I^+).$$

Now, let $\mathcal{I}$ be the ideal sheaf on the boundary of our formal model $X^\circ_{K_p K^p}$. Then we conclude

$$H^0(\mathcal{V}_{J_2}, I^+/p^n I^+) = \varinjlim_{K_p} \varinjlim_{S_{J_2}} H^0(X^\circ_{K^p K_p}, (\omega^{\mathrm{int}})^{\otimes k|J_2|} \otimes \mathcal{I}/p^n \mathcal{I}).$$

Here, we're replacing the cohomology for sections over $\mathcal{V}_{J_2}$ with sections over $X^\circ_{K^p K_p}$. The limit over S_{J_2} is allowing enough poles to make enough sense of this.

So it's enough to show that the action of $\mathbb{T}$ on this cohomology group for $X^\circ_{K^p K_p}$ factors through $\mathbb{T}_{\mathrm{cl}}$. Since ω^{int} is ample it's enough to show this for

$$H^0(X^\circ_{K^p K_p}, (\omega^{\mathrm{int}})^{\otimes k|J_2|} \otimes \mathcal{I}).$$

Since there's no torsion, it's enough to prove the statement after inverting p; but $\mathbb{T}_{\mathrm{cl}}$ factors through that group by definition.