Banach-Colmez spaces and coherent sheaves on the Fargues-Fontaine curve

(after L. Fargues)

Arthur-César Le Bras (Institut de Mathématiques de Jussieu)

The Banach-Colmez spaces and the fundamental curve of p-adic Hodge theory were introduced to give (new) proofs of conjectures of Fontaine in p-adic Hodge theory : the conjecture "weakly admissible implies admissible" and the conjecture "de Rham implies potentially semi-stable". Besides the fact that they help to solve the same questions, their theory leads to consider quite similar objects. The goal of this note is to try to explain why they are related.

Banach-Colmez spaces

In [Co], Colmez, inspired by some ideas of Fontaine, introduced objects now called Banach-Colmez spaces. These are defined as certain functors from the category of sympathetic algebras to the category of $\mathbf{Q}_p$-Banach spaces. More precisely, a *sympathetic algebra* A is a Banach C-algebra, endowed with its spectral norm $\|.\|_A$, connected and such that every $a \in A$, with $\|a-1\|_A < 1$ admits a p^{th}-root. A covariant functor (satisfying a continuity condition) from this category (in which morphisms are continuous morphisms of C-algebras) to the category of $\mathbf{Q}_p$-vector spaces is called a *Banach Space*. There exist two simple examples of such objects : for V a finite dimensional $\mathbf{Q}_p$-vector space, the functor V^{et} which associates V to any sympathetic A (with identity morphisms) and for W a finite-dimensional C-vector space, the functor W^{an} which associates $W \otimes_C A$ to A (with obvious morphisms). A sequence of Banach Spaces is said to be *exact* if it is so on A-points, for each sympathetic A. A *Banach-Colmez space* is then by definition a Banach Space E such that there exist two exact sequences of Banach Spaces :

$$0 \to V^{et} \to Y \to W^{an} \to 0,$$

$$0 \to V'^{et} \to Y \to E \to 0,$$

where V, V' are finite dimensional $\mathbf{Q}_p$-vector spaces, W is a finite dimensional C-vector space and Y is a Banach Space. Note that in particular Y is also a Banach-Colmez space, which is said to be *effective*. If such a presentation is given, one defines $\dim E = \dim_C W$ (the *dimension* on E) and $\mathrm{ht} E = \dim_{\mathbf{Q}_p} V - \dim_{\mathbf{Q}_p} V'$ (the *height* of E). For example, $\mathrm{ht}(\mathbf{Q}_p^{et}) = 1$, $\dim(\mathbf{Q}_p^{et}) = 0$; $\mathrm{ht}(C^{an}) = 0$, $\dim(C^{an}) = 1$. This gives two functions dim and ht on the category $\mathcal{BC}$ of Banach-Colmez spaces. Here is the main result of the theory.

Theorem 1 (Colmez)**.** *The category $\mathcal{BC}$ is abelian. The two functions*

$$\dim : \mathcal{BC} \to \mathbf{N} \quad ; \quad \mathrm{ht} : \mathcal{BC} \to \mathbf{Z}$$

are well-defined (i.e. do not depend on the choice of a presentation) and additive in exact sequences. Moreover, the functor of C-points on $\mathcal{BC}$ is exact and faithful.

Link with pro-étale sheaves and p-divisible groups

First, it could be possible to give an alternative definition of Banach-Colmez spaces. Let $\mathcal{C}$ be the category of locally noetherian adic spaces over C. The topos $\mathcal{T}$ is a fibred topos over $\mathcal{C}$, with fiber over $Y \in \mathcal{C}$ the *pro-étale topos* of Y (as defined by Scholze). An object of $\mathcal{T}$ is the datum for all $Y \in \mathcal{C}$ of a pro-étale sheaf $\mathcal{F}_Y$, and for every morphism $f : Y \to Y'$ in $\mathcal{C}$, of a morphism $u_f : f^*\mathcal{F}_{Y'} \to \mathcal{F}_Y$ of sheaves over Y, such that if $g : Y' \to Y''$ is another morphism, $u_f \circ f^* u_g = u_{g \circ f}$. Interesting objects $\mathcal{F}$ of $\mathcal{T}$ are those such that for every $Y \in \mathcal{C}$, $\mathcal{F}_Y$ is a $\widehat{\mathbf{Q}}_p$-sheaf. They form a full subcategory $\mathcal{G}$ of $\mathcal{T}$. If V is a finite dimensional $\mathbf{Q}_p$-vector space, the lisse sheaf $V^{et} = V \otimes \widehat{\mathbf{Q}}_p$ is in $\mathcal{G}$, as is $W^{an} = W \otimes_C \widehat{\mathcal{O}}_Y$ for every finite dimensional C-vector space W. Define an effective Banach-Colmez space to be an object of $\mathcal{G}$ extension of W^{an} by V^{et} for some V, W as before.
Note that an effective Banach-Colmez space in the above sense defines a Banach-Colmez space in the sense of Colmez. Indeed, let A be a sympathetic algebra and $\mathcal{F} \in \mathcal{G}$ an effective Banach-Colmez space. Write A as the p-adic completion of a direct limit of strongly noetherian Tate C-algebras A_i and let $E(A)$ be the p-adic completion of $\mathcal{F}(\varinjlim A_i)$. It does not depend on the choice of the presentation (A_i) of A, as it is the case for the sheaves V^{et} and W^{an}. The functor E is a Banach-Colmez space. This defines a functor $\mathcal{F} \mapsto E$ which is fully faithful (because of the existence of sympathetic closures of affinoid algebras) ; to show that it is an equivalence of categories, it suffices to check that $\mathrm{Ext}^1_{\mathcal{G}}(C^{an}, \mathbf{Q}_p^{et})$ is isomorphic to $C \simeq \mathrm{Ext}^1_{\mathcal{BC}^{\mathrm{eff}}}(C^{an}, \mathbf{Q}_p^{et})$. To do this one can copy the proof of Fargues in [Fa] in our context : he works over C, but in fact his computations go through without modification for every C-algebra A such that $\log : 1 + A^{\circ\circ} \to A$ is surjective, which is precisely a characterization of sympathetic algebras. Anyway the author does not know if a pretty definition of all Banach-Colmez spaces as sheaves can be given.
The results of Fargues and Scholze-Weinstein give another viewpoint on the subcategory $\mathcal{BC}^{\mathrm{eff}}$, which nicely explains why it appears naturally and how to construct elements in it. An effective Banach-Colmez space can be seen as a group object in the category of Stein spectral spaces over C, of the form $U(G) = \varprojlim_p G$, where G is a p-divisible group over $\mathcal{O}_C$ ($U(G)$ is the *universal cover* of G). The logarithm (period map) induces an exact sequence

$$0 \to V_p(G[p^\infty]) \to U(G)(C) \to \mathrm{Lie}\ G \to 0,$$

which is in fact obtained by taking the C-points of the exact sequence in $\mathcal{BC}$:

$$0 \to (V_p(G[p^\infty]))^{et} \to U(G) \to (\mathrm{Lie}\ G)^{an} \to 0.$$

Moreover, $U(G)$ depends only on the special fiber of G and $U(G)(C) = (M \otimes B^+)^{\varphi = p}$, where M is the Dieudonné module of G modulo p. This way we see that for every $0 \leq \lambda = d/h \leq 1$, $(B^+)^{\varphi^h = p^d}$ is the set of C-points of an element of $\mathcal{BC}^{\mathrm{eff}}$ and conversely each effective Banach-Colmez space is obtained by taking direct sums of these objects.
But even with this concrete definition of effective Banach-Colmez spaces, the whole category $\mathcal{BC}$ remains a bit mysterious and rather artificial. The goal of the next paragraphs is to find a more natural description of it.

The Fargues-Fontaine curve

The so-called *fundamental curve of p-adic Hodge theory* has been introduced by Fargues and Fontaine (see for example [FF]). Its definition was motivated by different questions of p-adic Hodge theory. Let $F = C^\flat$, a perfectoid field of characteristic p. The ring $B^{b,+} = W(\mathcal{O}_F)[1/p]$ is endowed with valuations v_r, for each $r > 0$:

$$v_r : \sum_{n \gg -\infty} [x_n] p^n \mapsto \inf_{n \in \mathbb{Z}} \{v(x_n) + nr\}.$$

The completion B^+ of $B^{b,+}$ with respect to the family $(v_r)_{r>0}$ is a Fréchet $\mathbf{Q}_p$-algebra, endowed with a Frobenius φ. Let $P_d = (B^+)^{\varphi = p^d}$ for $d \geq 0$ and

$$X = \mathrm{Proj}\ P := \mathrm{Proj}\left(\bigoplus_{d \geq 0} P_d\right).$$

This is a noetherian regular scheme of dimension 1. If $t \in P_1 - \{0\}$, $V^+(t)$ is a single point ∞, and the completed local ring $\widehat{\mathcal{O}}_{X,\infty}$ is isomorphic to B^+_{dR} (with residue field C). Moreover, $X \backslash \{\infty\}$ is the spectrum of the principal ring $B_e = (B^+[1/t])^{\varphi = \mathrm{Id}}$ (hence X is a kind of compactification of the affine scheme $\mathrm{Spec} B_e$). From the description of closed points of X, one can guess that X must be thought of as the quotient of the open unit disk by Frobenius and give a precise meaning to this assertion.
For $d \in \mathbf{Z}$, let $\mathcal{O}_X(d) = \widetilde{P[d]}$. This is a line bundle on the curve and one obtains all of them in this way (up to isomorphism). One can do a similar construction on $X \otimes \mathbf{Q}_{p^h}$ ($h > 0$) and get by push forward a vector bundle $\mathcal{O}_X(\lambda)$ on X, of rank h and degree d, for each rational $\lambda = d/h$, $(d,h) = 1$. It has slope $\mu(\mathcal{O}_X(\lambda)) = \lambda$. Fargues and Fontaine show that the semi-stable vector bundles of slope λ are precisely the direct sums of copies of $\mathcal{O}_X(\lambda)$ and that the Harder-Narasimhan filtration is split on X. The curve shares some similarities and some differences with $\mathbf{P}^1$: for instance $H^0(X, \mathcal{O}_X(\lambda)) \neq 0$ iff $\lambda \geq 0$ but $H^1(X, \mathcal{O}_X(\lambda)) \neq 0$ iff $\lambda < 0$ (example : $H^0(X, \mathcal{O}_X) = \mathbf{Q}_p$, $H^1(X, \mathcal{O}_X(-1)) = C/\mathbf{Q}_p$).

Torsion pairs and t-structures on derived categories

Let $\mathcal{A}$ be an abelian category. A *torsion pair* in $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories such that $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for every $T \in \mathcal{T}$ and $F \in \mathcal{F}$ and such that for every object $E \in \mathcal{A}$, there exists a short exact sequence

$$0 \to T \to E \to F \to 0,$$

with $T \in \mathcal{T}$, $F \in \mathcal{F}$. Let D be the bounded derived category of $\mathcal{A}$ and assume given a torsion pair $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$. The simple key proposition is the following.

Proposition 1. *The full subcategories*

$$\mathcal{A}^- := \{E \in D, H^i(E) = 0 \text{ for } i \neq 0, -1 \ ; H^{-1}(E) \in \mathcal{F}, H^0(E) \in \mathcal{T}\}$$

$$\mathcal{A}^+ := \{E \in D, H^i(E) = 0 \text{ for } i \neq 0, 1 \ ; H^1(E) \in \mathcal{T}, H^0(E) \in \mathcal{F}\}$$

are hearts of D.

In other words, there exist two new bounded t-structures on D with abelian cores $\mathcal{A}^+$ and $\mathcal{A}^-$. Note that $(\mathcal{T}, \mathcal{F}[1])$ is a torsion pair in $\mathcal{A}^-$, whence $(\mathcal{A}^-)^+ = \mathcal{A}$. For example, let Y be a smooth projective and $\mathcal{A}$ be the abelian category of coherent sheaves on Y. We define the slope $\mu(E)$ of a vector bundle E on Y to be the quotient of its degree by its rank and say that a torsion sheaf is of slope $+\infty$. For any coherent sheaf $\mathcal{F}$, there exists a unique increasing filtration (the *Harder-Narasimhan filtration*)

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F},$$

such that each $\mathcal{F}_{i+1}/\mathcal{F}_i$ is semi-stable μ_{i+1}, with $\mu_1 > \ldots \mu_n$. Fix a real number m, and define $\mathcal{T}_m$ (resp. $\mathcal{F}_m$) to be the fullsubcategory of coherent sheaves all of whose HN-filtration quotients have slope $\geq m$ (resp. $< m$). The previous remarks show that $(\mathcal{T}_m, \mathcal{F}_m)$ is indeed a torsion pair on $\mathcal{A}$. We denote $\mathcal{A}^{m,-}$ and $\mathcal{A}^{m,+}$ the new hearts obtained by tilting.

$\mathcal{BC}$ as an abelian core in $D^b(\mathrm{Coh}_X)$

Let now $\mathcal{A} = \mathrm{Coh}_X$ be the abelian category of coherent sheaves on the Fargues-Fontaine curve X, and $D = D^b(\mathrm{Coh}_X)$ its bounded derived category (here some care is needed but everything works fine since X is noetherian, proper and regular). The existence of the Harder-Narasimhan filtration allows us to define as before the abelian category $\mathrm{Coh}_X^{0,-}$.

Proposition 2. *The categories $\mathcal{BC}$ and $\mathrm{Coh}_X^{0,-}$ are equivalent.*

We can construct a functor $\mathrm{Coh}_X^{0,-} \to \mathcal{BC}$, which will realize the equivalence of categories. Let $\mathcal{F}$ be an object of $\mathrm{Coh}_X^{0,-}$. We should think to $\mathcal{F}$ as the pair $(H^{-1}(\mathcal{F}), H^0(\mathcal{F}))$, plus the datum of an element of $\mathrm{Ext}^1_{\mathrm{Coh}_X}(H^0(\mathcal{F}), H^{-1}(\mathcal{F})[1]) = \mathrm{Ext}^2_{\mathrm{Coh}_X}(H^0(\mathcal{F}), H^{-1}(\mathcal{F})) = 0$, as X is a curve. Hence $\mathcal{F}$ is just a pair $(\mathcal{F}^{-1}, \mathcal{F}^0)$. If A is any sympathetic C-algebra, we can define a version of the curve X_A living over X and hence sheaves $\mathcal{F}_A^{-1}$ and $\mathcal{F}_A^0$ on X_A by pull back. Let $E_{\mathcal{F}}(A) = H^0(X_A, \mathcal{F}_A^0) \oplus H^1(X, \mathcal{F}_A^{-1})$. The functor $E_{\mathcal{F}}$ is a Banach-Colmez space. Let us compute some examples. By this functor, the sheaf $\mathcal{O}_X$ is sent to $\mathbf{Q}_p^{et}$, $i_{\infty,*}C$ to C^{an}. Which sheaf corresponds to the Banach-Colmez space with C-points $C/\mathbf{Q}_p$? One has an exact sequence of coherent sheaves on X :

$$0 \to \mathcal{O}_X(-1) \overset{\times t}{\to} \mathcal{O}_X \to i_{\infty,*}C \to 0,$$

which we can rewrite in the derived category as an exact triangle

$$\mathcal{O}_X \to i_{\infty,*}C \to \mathcal{O}_X(-1)[1] \overset{+1}{\to}$$

From this we can guess that $\mathcal{O}_X(-1)[1]$ should correspond to $C/\mathbf{Q}_p$ and indeed $H^1(X, \mathcal{O}_X(-1)) = C/\mathbf{Q}_p$. Where does the proposition come from ? Note that the two categories $\mathcal{BC}$ and Coh_X have the same Grothendieck group ; therefore one can expect that they correspond to two t-structures on the same triangulated category. Obviously they are not directly equivalent, because the functor of global sections is not exact and faithful on Coh_X (because of bundles of negative slope) whereas the functor of C-points is exact and faithful on $\mathcal{BC}$. So to get this exactness we must look at complexes of sheaves : if for simplicity we try to work with complexes concentrated in two consecutive degrees, say -1 and 0, the spectral sequence for a complex $\mathcal{F}$

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(\mathcal{F})) \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{F})$$

naturally leads to the candidate $\mathrm{Coh}_X^{0,-}$.
Now we give a sketch of proof of the proposition. It is almost immediate to check that the functor $\mathcal{F} \to E_{\mathcal{F}}$ realizes an equivalence of categories between the subcategory $\mathrm{Coh}_X^{[0,1]}$ of vector bundles on X whose HN-quotients have slopes between 0 and 1 and $\mathcal{BC}^{\mathrm{eff}}$ (in fact we already gave a description of both these categories in terms of Fontaine's rings $(B^+)^{\varphi^h = p^d}$, $0 \leq h \leq d$, $(h,d) = 1$). To conclude we just have to show that Coh_X identifies with the category of pairs $(V' \otimes \mathcal{O}_X \hookrightarrow \mathcal{F})$, where V' is a finite dimensional $\mathbf{Q}_p$-vector space and $\mathcal{F} \in \mathrm{Coh}_X^{[0,1]}$, localized with respect to morphisms inducing isomorphisms on the cokernel. In other words, we just have to verify that any $\mathcal{F} \in \mathrm{Coh}_X$ can be written in an essentially unique way as the quotient of a vector bundle of slopes between 0 and 1 by a semi-stable vector bundle of slope 0, *in the category* $\mathrm{Coh}_X^{0,-}$. It suffices to do it for $\mathcal{F} = \mathcal{O}_X(\lambda)$, $\lambda < 0$ or $\lambda > 1$, and $\mathcal{F} = i_{\infty,*}B^+_{dR}/t^k B^+_{dR}$, $k > 0$. If $\lambda = d/h > 1$, by pull-back to X_h we can reduce to the case $h = 1$ and then use inductively the existence of an exact triangle (cf. [Co], 8.20) :

$$\mathcal{O}_X \to \mathcal{O}_X(1) \oplus \mathcal{O}_X(d-1) \to \mathcal{O}_X(d) \overset{+1}{\to},$$

If $k > 0$, the exact triangle

$$\mathcal{O}_X \to \mathcal{O}_X(k) \to i_{\infty,*}B^+_{dR}/t^k B^+_{dR} \overset{+1}{\to}$$

reduces to the above case. Finally, if $\lambda = d/h < 0$, considering the triangle

$$\mathcal{O}_X^h \to i_{\infty,*}B^+_{dR}/t^{-d} B^+_{dR} \to \mathcal{O}_X(\lambda)[1] \overset{+1}{\to}$$

induced by multiplication by an element in $(B^+)^{\varphi^h = p^{-d}}$ again puts us in a known case.

Some corollaries

Taking for granted the properties of the fundamental curve, one can recover some results of Colmez on the category $\mathcal{BC}$. In the above, the only thing we used in his paper is the definition of $\mathcal{BC}$ (and a key computation of Ext^1 for the identification with a category of sheaves). Therefore the proposition 2 proves that the category $\mathcal{BC}$ is abelian and that the functor of C-points is exact and faithful. Moreover, as the curve X only depends on $F = C^\flat$ and not on C, the same is true for $\mathcal{BC}$. Conversely, the category $\mathcal{BC}$ allows to recover (abstractly) the curve X : indeed $\mathrm{Coh}_X = (\mathcal{BC})^{0,+}$ and by a theorem of Gabriel, a regular scheme can be reconstructed up to isomorphism from the abelian category of coherent sheaves on it.
Let us return to the notations of the end of the fourth subsection. It is not hard to see that the category $\mathcal{A}^{m,-}$ is endowed with a Harder-Narasimhan filtration with respect to the slope function

$$\mu^{m,-} = \frac{-\mathrm{rk}}{\deg - m\ \mathrm{rk}}.$$

Here we used the fact that degree and rank are additive functions. In particular, we have a Harder-Narasimhan filtration on $\mathcal{BC}$, already studied by Plût, for the slope function

$$\mu_{\mathcal{BC}} = \frac{-\mathrm{rk}}{\deg} = -\frac{\mathrm{ht}}{\dim}.$$

References

[Co] P. Colmez. Espaces de Banach de dimension finie. *J. Inst. Math. Jussieu 1* (2002), 331-439.

[Fa] L. Fargues. Groupes analytiques rigides p-divisibles. To appear in *Math. Annalen*.

[FF] L. Fargues et J.-M. Fontaine. Vector bundles on curves and p-adic Hodge theory. To appear in the Proceedings of the Symposium ESPRC "Automorphic forms and Galois representations" (Durham, 2011).

[Pl] J. Plût. Espaces de Banach analytiques p-adiques et espaces de Banach-Colmez. PhD Thesis (2009).