Three dimensions of course design for preservice secondary teachers

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UA Secondary Mathematics Education Program

Courses for prospective mathematics teachers at Arizona:

- Math 330: Topics in Geometry
- Math 315: Number Theory and Modern Algebra
- Math 361: Statistics for Teaching
- Math 407: Synthesis of Mathematical Concepts

- Math 406A: Curriculum and Assessment in Mathematics
- Math 406B: Methods of Teaching Secondary Mathematics
- Math 494C: Student Teaching

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. . . could be of real benefit to secondary mathematics teachers?

. . . has the potential to stick with them through their field experience and into their first year as full-fledged teachers?

Three Dimensions for the Capstone Course

In my capstone course, I wanted students to

- **Look deep:** Develop a thorough understanding of key concepts in high school mathematics: what concepts mean, what certain procedures signify and how they work
- **Look wide:** Develop perspective on the subject of mathematics as a whole, making connections among different ideas and representations
- **Look far:** Develop the mathematical practices and persistence necessary to apply K–12 ideas to challenging problems in mathematics and the real world

Structure of the Capstone Course

I divided the course into several "storylines," with each storyline taking two to three weeks of class. Some examples:

- The isomorphism between $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$
- Quadratic equations: algebraic and geometric points of view
- Modeling: using unseen quantities and unseen structure

Components of a Storyline

I tried to organize each storyline so that it did several things:

- Developed deeper understanding of a key mathematical idea
- Connected different mathematical ideas and said something interesting about mathematics as a discipline
- Showed how high school mathematics applies in novel and unexpected situations

In addition, I wanted each storyline to say something (at least implicitly) about mathematics teaching or the development of mathematical understanding.

An example of the design process for a storyline

"Quadratic equations: algebraic and geometric points of view"

The Genesis

. . . However, there is more to know about completing the square than how to carry out the technique: it reduces every single-variable quadratic equation to an equation of the form $x^2 = k$*, and thereby leads to a general formula for the solution of a single-variable quadratic equation; it generalizes to a method of eliminating the next to highest order term in higher order equations; it allows one to translate the graph of every quadratic function so that its vertex is at the origin, and thereby allows one to show that all such graphs are similar; and it provides an important step in simplifying quadratic equations in two variables. . .*

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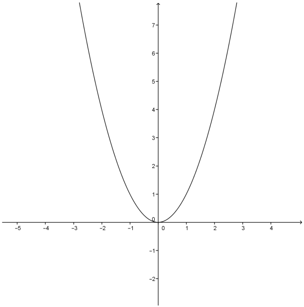
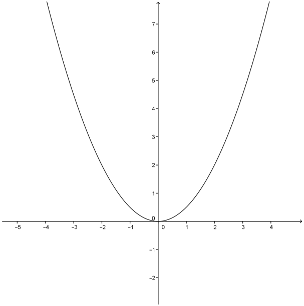
Here's the parabola $y = x^2$:



We can dilate this by a factor of 2 from the origin by replacing x and y in the equation with $(x/2)$ and $(y/2)$, respectively:

This produces the equation $y/2 = (x/2)^2$, or equivalently, $y = x^2/2$:



The Mathematical Potential

The task of showing that the graphs of any two quadratic functions are similar has terrific **mathematical potential** for preservice teachers:

- Understanding similarity in terms of dilations and rigid motions
- Specifying a sequence of transformations that will map one figure onto another
- Expressing these transformations both geometrically and algebraically
- Generalizing this process so that it works for the graphs of any two quadratic functions

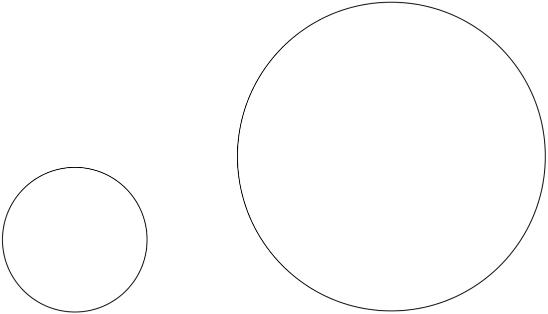
I tell students that the day's lesson is going to be on **similarity**. I start by asking students to write a definition of what it means for two figures to be similar.

Many students are able to do this correctly because the definition of similarity in terms of dilations is a point of emphasis in Math 330 (Geometry).

I then say we're going to play a game called "Are They Similar?". I project the following five images fairly quickly, and ask groups to agree on whether each pair of figures is similar:

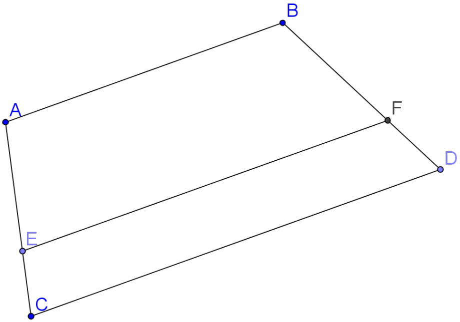
Are They Similar?

Slide 1



Are They Similar?

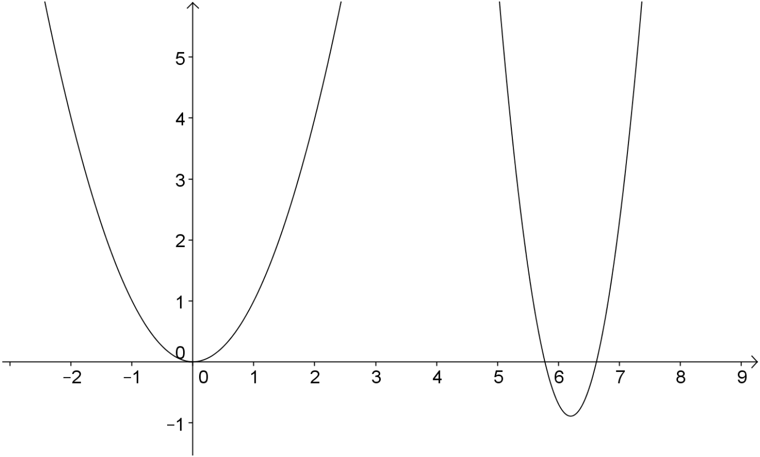
Slide 2



Trapezoids ABDC and ABFE

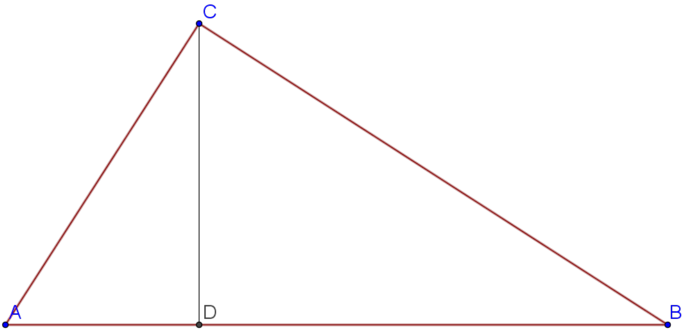
Are They Similar?

Slide 3



Are They Similar?

Slide 4



Right triangles ΔABC *and* ΔACD

Are They Similar?

Slide 5

I then ask students to share their decisions with the class and justify them.

On Slide 3 (the two parabolas), about half said "not similar." Most of the rest were unsure and/or suspected that I was up to something.

I then show a **GeoGebra demo** in which a parabola is dilated:

The Presentation

Once students are convinced that parabolas of different “widths” can be similar, I ask them to prove that the graphs of any two quadratic functions are similar. This is challenging for at least a couple of reasons:

- Some unpacking of the task needed (*e.g.*, how do we prove something about the graphs of any two quadratic functions?)
- Need to figure out how to represent transformations algebraically, particularly dilation

An Unexpected Turn

Out of students' work on this task, a new idea appears: thinking about a parabola in terms of its focus and directrix.

So we spend some time showing that the graph of a quadratic function has a focus and directrix. Some students then use this characterization of parabola to show that any two parabolas are similar (not just the ones that are graphs of equations $y = ax^2 + bx + c$).

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A bit of trickiness

A Detour

In the process of having students work on this task, I've noticed that some additional work with transformations of graphs is needed. I decide that we're going to lay a stronger conceptual foundation for why translations and stretches/shrinks of graphs work the way they do.

The group activity "Point Checker"

A Detour

Problem 1 – Checker	Problem 1 – Transformer
Let A be the graph of the equation $$x^2 - y^2 + 4y = 1.$$	Let B be the graph obtained by reflecting A across the x-axis.
Problem 1 – Point Person	**Problem 1 – Generalizer**
Are the following points on B? (2, 3) (1, -4) (0, -1)	Write a rule for checking whether a point (x, y) is on B.

A Detour

Most of the equations given are difficult to graph by hand, forcing students to think more at the abstract level: what would it mean for a point to be on the transformed graph?

The transformations start out easy (reflection across an axis) and progress toward ones for which students aren't likely to have seen a procedure already (rotate 90° counterclockwise around the origin).

A Detour

By the end of the activity, I want students to realize that

> *If T is a transformation and A is the graph of an equation, then a point (x, y) is on $T(A)$ if and only if $T^{-1}(x, y)$ is on the graph A.*

That's why we end up "subtracting a" in an equation when we're trying to move the graph a units to the right; "dividing by k" if we're trying to stretch by a factor of k; *etc.*

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(*Getting beyond "That's one of the places in math where the rules are backwards": thinking about what procedures signify*)

Looking Toward the Horizon

Now that students have a more general understanding of how to represent transformations of a graph algebraically, it's time to see what new mathematical ideas we can develop with this.

So I remind students that points in $\mathbb{R}^2$ can be rotated about the origin using the matrix

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...and then ask them to use this to rotate the hyperbola $x^2 - y^2 = 1$ counterclockwise about the origin by 45°.

Looking Toward the Horizon

This leads to the equation

$$\left(\frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}y\right)^2-\left(\frac{1}{\sqrt{2}}x-\frac{1}{\sqrt{2}}y\right)^2=1,$$

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After students have figured this out by hand, I show them a **GeoGebra demo**:

Connections to Algebraic Structure

To illustrate a connection between this topic and algebraic strructure, I pose the following homework problem:

Use GeoGebra to investigate the effect of changing the parameter a on the graph of the equation

$$3x^2 + axy + 2y^2 - 7x + 4y + 2 = 0.$$

At one value of a, the graph of the equation is not a figure that is usually considered a conic section. Give an algebraic explanation for why this figure appears. Can this figure be thought of as a "conic section"?

Connections to Algebraic Structure

What's going on at $a = -7$?

The graph seems to consist of the lines $y = 3x - 1$ and $y = \frac{1}{2}x - 1$ – equivalently, $3x - y - 1 = 0$ and $x - 2y - 2 = 0$.

This graph would then be produced by the equation $(3x - y - 1)(x - 2y - 2) = 0\ldots$ which works out to

$$3x^2 - 7xy + 2y^2 - 7x + 4y + 2 = 0.$$

Some Remarks on Design

Throughout the storyline, I aim to address three "dimensions" of mathematical knowledge:

- **Look deep:** Careful understanding of how algebraic transformations correspond to graph transformations; understanding of similarity in terms of rigid transformations and dilations
- **Look wide:** Connecting algebraic and geometric views of a parabola (and other conic sections); connecting transformations and coordinate geometry
- **Look far:** Developing algebraic ways of thinking about transformations other than "the usual ones"; *e.g.*, rotating conic sections

Some Remarks on Design

I'm also trying to make some points about the subject of mathematics; in particular the role of definitions:

- Sometimes an informal definition, while it can help ground our understanding, is not sufficiently precise or powerful (similarity as "same shape, different size" or as "having all the same angles")
- When we have more than one definition of the same concept (*e.g.*, parabola), we ought to investigate the relationship between them

Some Remarks on Design

Also, I'm trying to make some points about the teaching and learning of mathematics:

- Technology can do more than simply make mathematics flashier – it can allow us to investigate problems by trying many cases or possible solutions in rapid succession
- On the other hand, sometimes the best way to understand a difficult concept is to step back and think about it in the abstract, without the interference that can come from familiar examples

Techniques for Task Design

In order to push math majors' understanding of K–12 mathematics deeper/wider/farther, I use some task design techniques to neutralize prior, incomplete knowledge that might get in the way:

- **Jamming:** Pose a mathematical task in which the underlying concepts are essential, but the procedure cannot be used (*e.g.*, due to insufficient information) –
- **Crashing:** Pose a mathematical task in which mindless execution of the procedure is possible but likely to lead to a wrong answer –

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- **Crashing:** Pose a mathematical task in which mindless execution of the procedure is possible but likely to lead to a wrong answer – "Are They Similar?"

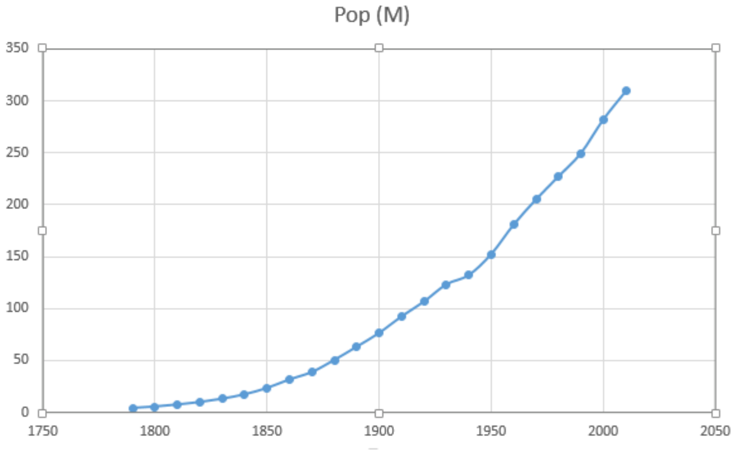
Snapshot of Another Storyline

An example of how the three "dimensions" of content knowledge are instantiated in another storyline:

Storyline: The isomorphism between $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$

- **Look deep:** What it means for two systems of arithmetic (or parts of arithmetic) to have the same structure; why we define $x^{1/n} = \sqrt[n]{x}$,
- **Look wide:** Linear functions as increasing by the same amount over each unit interval *vs.* exponential functions as increasing by the same factor over each unit interval; transforming an exponential function into a linear function using the logarithm
- **Look far:** How many digits are in 867^{5309}?; Benford's law – why do so many numbers have leading digit 1?

The Logarithm Transformation



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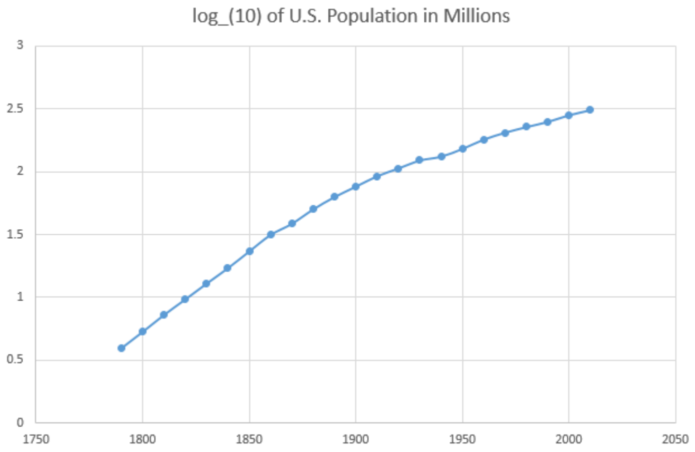

A Shortcoming

I really wanted to spend more time in my course developing aspects of specialized content knowledge for teaching:

- Dissecting a big mathematical idea, identifying the key developmental understandings that compose that idea ("knowing what you know")
- Selecting examples and tasks that help develop students' understanding of a mathematical idea
- Analyzing a task or student understanding and deciding whether it reaches the full depth of a standard or learning objective

We got at some of this by doing the PreService Teacher Task Study Project (PST-TSP), but want to do more.

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Might be helpful, for example, to have students develop examples that show how certain definitions don't work well, or how certain procedures can be problematic in the absence of understanding.

One Main Idea

Students in our mathematics major get a lot of experience with mathematics at the level of calculus and beyond. This experience is valuable!

My goal is to convince students that the mathematics of grades K–12 is deeply interesting – and that the habits they have developed as mathematics majors, when brought to bear on K–12 mathematics, can open up new understanding, add coherence, and reveal the power of simple ideas.

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(And we should share those habits with K–12 students!)

Thank you!

For Dropbox access to my slides and some of my course materials, e-mail me:
cpatterson@math.arizona.edu