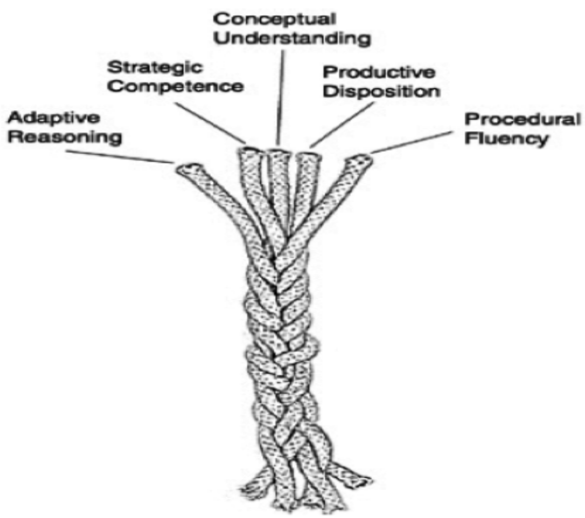

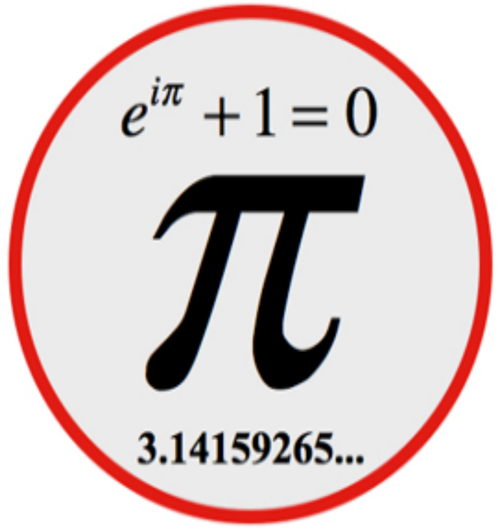

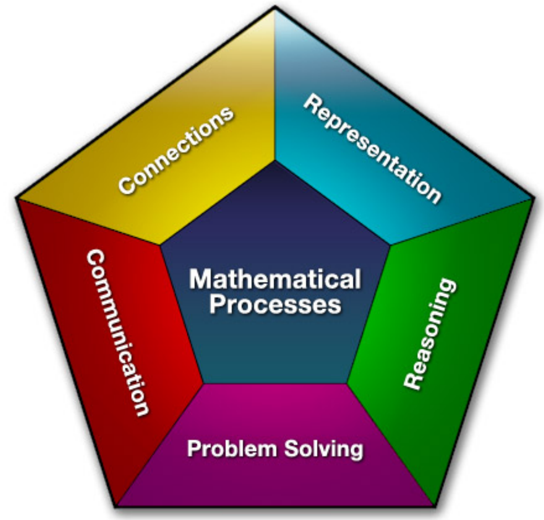

A Course on Mathematical Connections and Connected Mathematical Thinking

(for both pre-service teachers and math/science majors)

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Description of the Course
It is Problem-Based, Interactive

- **Rationale**: To foster mathematical practices, especially making mathematical connections
- **Instruments**: Problems that invite/require seeing/ making connections
- **Topics**: Around the arithmetic, algebra, and geometry of the number line

Finding/Making/Using Mathematical Connections & Structure

Mathematical Connections and Cognition

- The school (and even college) mathematics curriculum efficiently organizes mathematics into distinct subjects, but students, even when academically successful, often lose awareness of <u>the unity and connectivity of mathematical ideas</u> across domains.

- Perhaps this is because modest amounts of more advanced theory are needed to see these cross-domain connections clearly. Learning to make and see such connections is one of the aims of the course.

- The cognitive literature on learning suggests that networks of connections is characteristic of deep understanding and of high problem solving skills.

Making Mathematical Connections
A Core Practice

- **The feature of mathematics that is least visible in the school curriculum is its conceptual unity and coherence.**
- To cultivate more flexible mathematical thinking, some have proposed the use of problems with multiple solutions, and multiple solution strategies. This is valuable, but . . . .
- To emphasize connections (and transfer) I have tried, more broadly, to design "**cross domain problems**," problems whose resolution needs to draw resources from different mathematical topic areas (arithmetic, algebra, geometry, combinatorics, etc.)
- The curriculum typically isolates each domain into a conceptual and methodological box. So I am seeking to prompt students to "think outside the(se) box(es)."

The Nature and Importance of Connections

- **Types of connections**:
  - Connections of mathematics with "real world" situations and science. (Modeling).
  - Connections among different mathematical concepts, topics. (Cross domain)
  - Connections among different mathematical problems or situations (Common structure)
- **Cognition**
  - Expert vs novice problem solvers: Knowledge is highly connected (networked)
  - Transfer: Connected thinking enables recognition of common structure across diverse mathematical contexts.
  - Memory: Networked knowledge is deeper, more efficient, and high leverage in holding complex bodies of information.

Mathematical Practice

- Mathematical practice: Disciplinary mathematical practice is generally acknowledged to involve two modes of activity – *problem solving and theory building*. Theory building is a high level form of connection making and conceptual unification.
- While problem solving has a significant presence in the school curriculum, it is less clear whether there is some credible, school appropriate, version of theory building.
- The course offers one learning activity that could serve as a kind of bridge between problem solving and theory building.

Some Illustrative Cross Domain Problems

Example

- **What is the Mathematical Domain of this Problem?**

- Find all (real valued) functions $f(x)$ of a real variable x that satisfy the condition:

$$(*) \qquad |f(x) - f(y)| \quad = \quad |x - y|$$

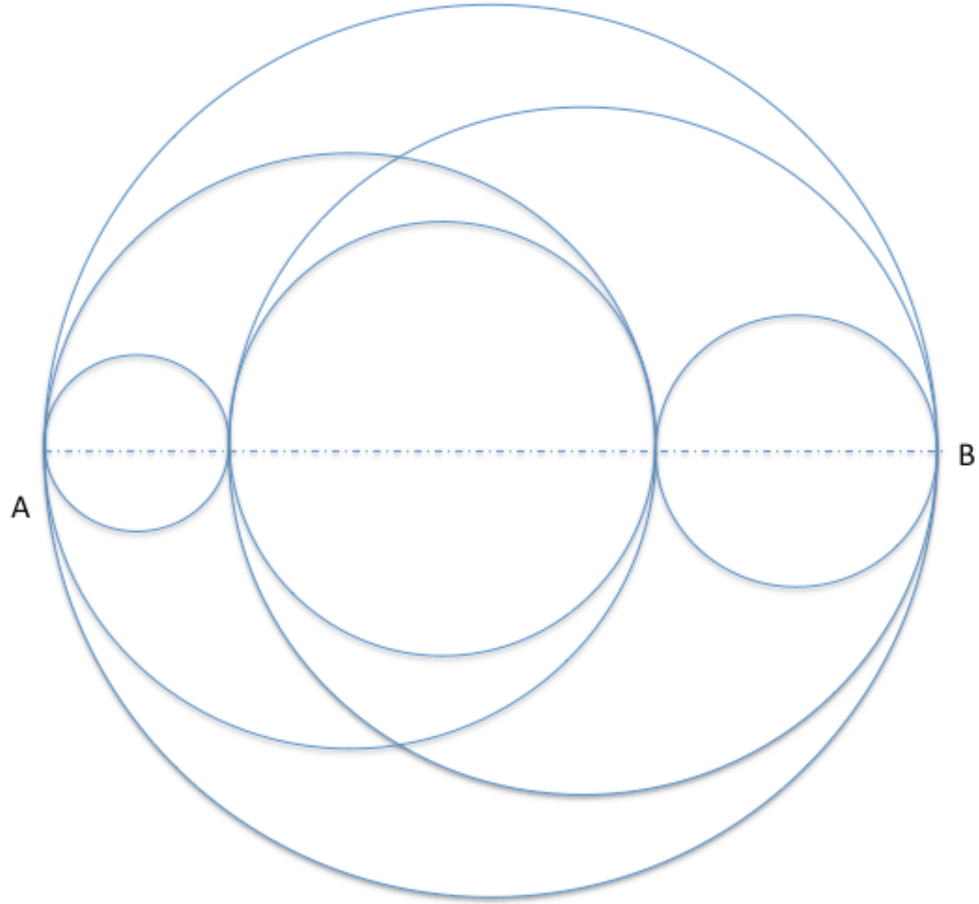
  for all real numbers x and y.

- Pre-calculus? Algebra? Geometry?

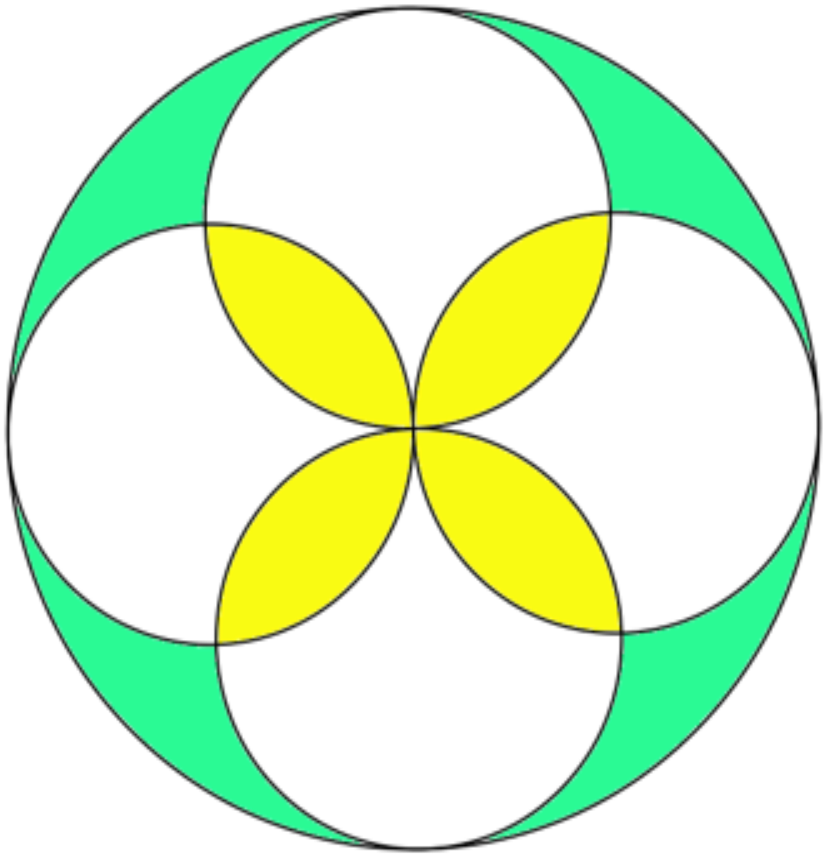
- Isometries of the line: Geometry

Traveling in Circles

Circle Park has a network of circular trails for cyclists to use (see Figure below). The trails have bridges so that they meet only along the diameter AB. Which is the shortest way to travel from A to B using this network of circular paths?



Which area is more: The green or the yellow? Explain your answer.



Where is f continuous?

- $f(x) = 0$ for x irrational
- $f(x) = 1/q$ for $x = p/q$, rational, in reduced form

- Key Lemma: If a sequence of rational numbers approaches an irrational number, their denominators approach ∞.

Topics

ToC of a Proposed Text

Place Value
and
Modular Congruence

Division with Remainder (DwR)
(A cornerstone of the course)

- a, b real numbers, $b \neq 0$.
- $a = qb + r$ (uniquely) q in $\mathbf{Z}$, $0 \leq r < |b|$
  $= q_b(a)b + r_b(a)$
  - The case $b = 1$: $[a] = q_1(a)$, "integer part" $\langle a \rangle = r_1(a)$ "fractional part"
  - $q_b(a) = [a/b]$ $r_b(a)/b = \langle a/b \rangle$
- $b \mid a \Leftrightarrow r_b(a) = 0$

- Congruence mod m:
  - $a =_m b$ means $m \mid (b - a)$
  - $=_m$ preserves sums and (for integers) products.

Base-b expansions

- b an integer > 1 Base-b digits: {0, 1, . . . , b-1}
- a a real number ≥ 0

- **a = $\sum_h d_h(a)b^h$** **$d_h(a) = r_b([ab^{-h}])$**
- $d_h(a)$ = 0 for h >> 0.

For a ≥ 1:

- $\delta(a)$ (or $\delta_b(a)$) $=_{Def}$ 1 + max $\{h \mid d_h(a) > 0\}$
- Order of magnitude of a = $b^{\delta(a)-1}$
- For an integer a,
  - $\delta(a)$ = the number of significant digits of a
  - $\delta(aa') \leq \delta(a) + \delta(a')$ so $\delta(a^e) \leq e \cdot \delta(a)$

Base-2:
How Many Multiplications Needed to Calculate N^E ?

Using iterated squaring, show that you need at most

$$n(s - 1)$$

multiplications, where $n = \delta_2(N) - 1$, and s is the sum of the base-2 digits of E.

(Much smaller than $E - 1$.)

Base-1,000

What is the base-1000 representation of

N = 48,574,623,791,105 ?

Base-10: "Casting Out Nines" "Divisibility Test," OR Fast Track to the Remainder

- $S(N) =_{Def}$ the sum of the digits of N

- **School math**: $9|N \quad \Leftrightarrow \quad 9|S(N)$

- **Math**: $S(N) \;=_9\; N \qquad (10 \;=_9\; 1)$
- S(N) is <u>much smaller</u> than N
- $S(N) \;\leq\; 9 \bullet \delta(N)$

Problem

Let

A	=	4444^{4444}
B	=	S(A)
C	=	S(B)
D	=	S(C)

What is D?

Base-b Expansion of N/D (proper, reduced)

- **$d_{-h}(N/D) = q_D(r_D(Nb^{h-1})b)$**
- There exist integers (t, p), $t \geq 0$, $p > 0$, such that $d_{-(h+p)}(N/D) = d_{-h}(N/D)$ for all $h > t$
- The least values of (t, p) are determined as follows:
- Write $D = D_0 D_1$ so that all primes dividing D_0 also divide b, and $\gcd(D_1, b) = 1$. Then:
- t = the least t such that $b^t =_{D_0} 0$
  = "wait time" (order of nilpotency of b mod D_0) and
- p = the least p such that $b^p =_{D_1} 1$
  = "period" (order of b in the group $(\mathbf{Z}/\mathbf{Z}D_1)^x$)

(Real) Additive Groups

A set A of real numbers is called an

- <u>**Additive group**</u> if:

(0) 0 is in A

(+) a, b in A => a + b is in A

(-) a in A => -a is in A

- A is **discrete** if 0 is "isolated" in A.
- In that case there exists an $e > 0$ such that $|a - b| \geq e$ for all $a \neq b$ in A. (A is *uniformly discrete*)

Examples & Non-Examples

$\mathbf{Q}$ = the set of all rational numbers

$\mathbf{Q}_{sq}$ = {squares of rational numbers}

$\mathbf{Q}_{0dd}$ = {rational numbers with an odd denominator}

$\mathbf{Q}_{sqd}$ = {rational numbers with a square denominator}

$\mathbf{Q}_{sfd}$ = {rational numbers with square free denominator}

In the following examples, d is a fixed integer > 0.

$\mathbf{Q}_{\leq d}$ = {rational numbers with denominator ≤ d}

$\mathbf{Q}_{|d}$ = {rational numbers with denominator that divides d}

$\mathbf{Q}_{d^*}$ = {rational numbers with denominator dividing some power of d}

Numbers with finite decimal expansion (= $\mathbf{Q}_{10^*}$)

$\mathbf{R}_{<0}$ = {real numbers < 0}

Per(f)

- Let $f(x)$ be a function of a real variable.

- Call a real number p a <u>period</u> of f if

  $f(x+p) \quad = \quad f(x) \quad$ for all x.

- Let

  $Per(f) \quad = \quad$ the set of <u>all</u> periods of f

- Show that $Per(f)$ is an additive group.

Structure Theory

- **Easy applications of DwR** !!
- **TH 1**. An additive group is either discrete or dense (DDD)

- **TH 2**. Discrete groups are cyclic: $A = \mathbf{Z}a$ (DGC)

- **TH 3**. $\mathbf{Z}a + \mathbf{Z}b$ is discrete iff a and b are commensurable.
  - Cor. $\mathbf{Z} + \mathbf{Z}\sqrt{2}$ is dense in $\mathbf{R}$.

- Then $\mathbf{Z}a + \mathbf{Z}b = \mathbf{Z}d$, and $\mathbf{Z}a \wedge \mathbf{Z}b = \mathbf{Z}m$,
  - $d = \gcd(a, b)$, $m = \operatorname{lcm}(a, b)$

- **Real multiplicative groups:**
- Examples; torsion $\{\pm 1\}$; sgn and $|\bullet|$ homomorphisms
- What should discreteness mean?
- Structure of discrete multiplicative groups (exp and log isomorphisms)

Additive Semi-groups

- $\mathbf{N}(a, b) = \mathbf{N}a + \mathbf{N}b$

- **Theorem.** If a and b are relatively prime natural numbers, then the largest integer not in **N**(a, b) is
  $ab - a - b = (a-1)(b-1) - 1$

- For three or more numbers there is no known formula (Frobenius number)

Combinatorics

n-Choose-d

- ${}_nC_d$ $=_{Def}$ the number of d-element subsets of an n-element set, say [n] = {1, 2, . . . , n}.
- ${}_nC_d$ = n(n-1)• • •(n-d+1) / d!
  - n(n-1)• • •(n-d+1) = the number of <u>ordered</u> d-tuples of distinct elements of [n]

    (product of d consecutive integers)
  - **Cor**. n! = the number of orderings (permutations) of [n]
  - **Cor**. d! = the number of different orderings of d distinct elements.

<u>Mental note</u> about n(n-1)• • •(n-d+1)/d! :

"Hey, I'm an integer!"

"Funny, you don't look like it."

The Next Class

Show that:

A product of d consecutive integers
is always divisible by d!

Relatively Prime Factorizations

- Let N be an integer > 0.
- Show that the number of factors d of N such that gcd(d, N/d) = 1 is 2^r, where r is the number of distinct prime divisors of N.

OR

- How many factors d of N are there such that gcd(d, N/d) = 1?

(cross domain)

Prime Power Factorization

- $N \;=\; q_1 \cdot q_2 \cdot \cdots q_r$
- $q_j \;=\;$ power of a prime p_j
- $p_1, p_2, \ldots, p_r$ distinct
- $d \;=\;$ a product of a subset of $\{q_1, q_2, \ldots, q_r\}$

Crossing a Bridge to:

- Combinatorics: # subsets of $\{q_1, q_2, \ldots, q_r\} = 2^r$

Discrete Calculus

Sequences as Functions

- $\mathbf{F} = \mathbf{R}^{\mathbf{N}} = \{\text{functions } f\colon \mathbf{N} \longrightarrow \mathbf{R}\}$
- Same thing as an infinite sequence

  $(f(0), f(1), f(2), f(3), \ldots, f(n), \ldots)$

  of real numbers.
- Commutative ring (containing **R**: constant functions)

Discrete Calculus

- Δf ("derivative") and Sf ("integral")

$(\Delta f)(x) = f(x+1) - f(x)$

and $(Sf)(x) = f(0) + f(1) + \ldots + f(x-1)$

(Note: $(Sf)(0) = 0$)

Properties:

(a) $\Delta f = 0 \Leftrightarrow$ f is constant.

(b) Hence, $\Delta f = \Delta g \Leftrightarrow g = f +$ constant.

(c) $\Delta(f \bullet g)(x) = (\Delta f)(x) \bullet g(x+1) + f(x) \bullet (\Delta g)(x)$

(Product rule)

(d) If $f(x) = x^n$ then

$$(\Delta f)(x) = (x+1)^n - x^n = \Sigma_{0 \leq d < n} \; {}_nC_d \bullet x^d$$

The Fundamental Theorem of Discrete Calculus (FTDC)

For any f in **F**, we have:

$$\Delta(Sf) = f, \quad \text{and}$$

$$S(\Delta f) = f - f(0)$$

("constant of integration")

The binomial polynomials

For each integer $d \geq 0$ we define the polynomial

- $B_d(x) \;=\; "{}_xC_d" \;=\; x(x-1)\cdots(x-d+1)/d!$

  (We agree to put $B_d = 0$ for $d < 0$.)

- $B_0(x) = 1, \quad B_1(x) = x, \quad B_2(x) = x(x-1)/2$
- $\deg(B_d) = d$, and its leading term is $x^d/d!$
- $B_d(x) = 0$ for $x = 0, 1, \ldots, d-1$ (d roots)

Moreover:

- $B_d(x)$ is an integer whenever x is an integer.
- **(Pascal)** $\boldsymbol{\Delta B_d = B_{d-1}}$
- $SB_d = B_{d+1}$
- $\{ B_d \mid d \geq 0\}$ is a basis for $\mathbf{R}[x]$

x^d as a linear combination of Binomials

d = 0: $x^0 = 1 = B_0$

d = 1: $x^1 = x = B_1$

d = 2: $x^2 = B_1 + 2B_2$

d = 3: $x^3 = B_1 + 6B_2 + 6B_3$

d = 4: $x^4 = B_1 + 14B_2 + 36B_3 + 24B_4$

When $f = a_0B_0 + a_1B_1 + ... + a_dB_d$

$\Delta f = a_1B_0 + a_2B_1 + ... + a_dB_{d-1}$

and

$Sf = a_0B_1 + a_1B_2 + ... + a_dB_{d+1}$

(Calculus couldn't be easier!)

Results.

- If f is a polynomial of degree d, then Sf is a polynomial of degree d+1.
- For f in **F**, if Δf is a polynomial of degree d – 1 then f is a polynomial of degree d.
- f(x) is an integer for all integers x iff all a_j are integers.

Sums of d^{th} powers of coneseuctive integers

For $d \geq 0$, Put $S_d(x) = 0^d + 1^d + 2^d + \ldots + (x-1)^d$

Observation: If we put $P_d(x) = x^d$ then $S_d = SP_d$

Writing P_d as a linear combination of binomials then we get an easy formula for S_d.

$d = 1$: $S_1(x) = (SB_1)(x) = B_2(x) = x(x-1)/2$

$d = 2$; $P_2 = B_1 + 2B_2$:

$$\begin{aligned} S_2(x) &= SP_2 = B_2(x) + 2B_3(x) \\ &= x(x-1)/2 + 2x(x-1)(x-2)/6 \\ &= [x(x-1)/6][3 + 2(x-2)] = (x-1)x(2x-1)/6 \end{aligned}$$

$d = 3$; $P_3 = B_1 + 6B_2 + 6B_3$ so

$$\begin{aligned} S_3(x) &= B_2(x) + 6B_3(x) + 6B_4(x) \\ &= x(x-1)/2 + x(x-1)(x-2) + x(x-1)(x-2)(x-3)/4 \\ &= [x(x-1)/4][2 + 4(x-2) + (x-2)(x-3)] \\ &= [x(x-1)/4][x^2 - x] \\ &= [(x^2 - x)/2]^2 = S_1(x)^2 \end{aligned}$$

Discrete Calculus as a Nexus of Mathematical Themes

- Numerical sequences: Looking for growth patterns (linear, quadratic, . . .) using difference methods

- The formal analogy with calculus

- Combinatorics and polynomial algebra: binomial coefficients (Pascal); the Binomial Theorem

- Power sums: $S_d(n) = 0^d + 1^d + 2^d + , , , + (n-1)^d$ as discrete integrals. S_d is a polynomial of degree d+1.

- Linear algebra: Expressing the x^d $(d \geq 0)$ in terms of the new basis $(B_d(x))_{d\geq 0}$. Integer values at integers.

Common Structure Problem Sets

An instructional design to bridge problem solving and "theory building": **Common Structure Problem Sets**

0. A set of 5-12 problems, with a multi-part <u>assignment</u>:
    1. *Solve* the problems.
    2. *Find, articulate, and demonstrate* ***a*** *mathematical structure common* to all of the problems.
    3. In what ways are the problems *different*?

The 3-Permutation CSPS

1. What are all three-digit numbers that you can make using each of the digits 1, 2, and 3, and using each digit only once?

2. Angel, Barbara, and Clara run a race. Assuming there is no tie, what are all possible outcomes of the race (first, second, third)?

3. You are watching Angel, Barbara, and Clara playing on a merry-go-round. As the merry-go-round spins, what are all the different ways that you see all three of them in order, from left to right?

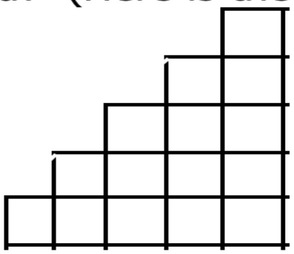
4. In a 3 x 3 grid square, color three of the (unit) squares blue, in such a way that there is at most one blue square in each row and in each column. What are all ways of doing this?

5. Find all of the symmetries(*) of an equilateral triangle

Hand-shakes CSPS (n-choose-2)

(a) (Consecutive sums) Find the sum of the integers from 1 to n-1: 1+2+3+. . . +(n-1

(b) (Side area of a staircase) In an (n-1)-step staircase made from unit cubes, how many cubes are needed? (Here is the 5-step staircase.)



(c) How many points (x, y) in the plane are there such that x and y are integers and $0 < y < x < n$?

(d) (Hand shakes) In a group of n people, each person shakes hands with each of the others. How many hand-shakes occur?

(e) (Two person delegation) From a group of n voters, how many ways are there to select a two-person delegation to attend the party caucus?

(f) (Diagonals of a polygon) In a convex polygon with n vertices how many diagonals (lines joining non-adjacent vertices) are there?

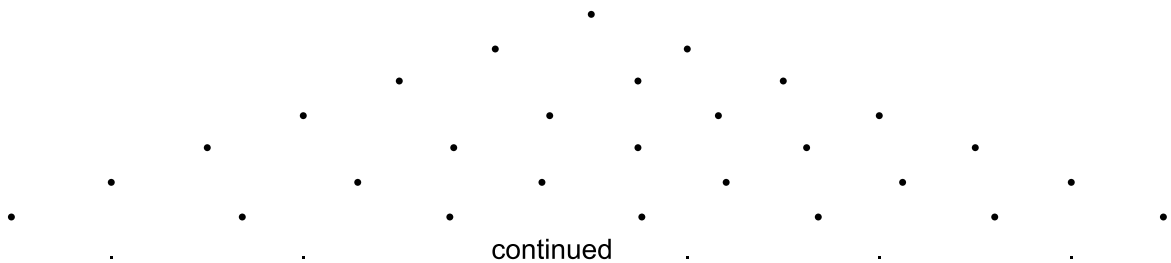
(g) (Pizza pieces) Using a pizza knife, what is the largest number of pieces of pizza that can be made with n-1 cuts? (Equivalently, what is the largest number of regions of the plane that can be carved out by n-1 lines?)

(h) (Recursion) Let $f(n)$ be a function of an integer variable $n \geq 0$ such that $f(1) = 2$, and $f(n) = f(n-1) + n$ for all $n \geq 1$. What is $f(n-1)$?

Pascal CSPS (14-choose-4)

1. (**Taxi cab geometry**). A taxi wants to drive (efficiently) from one corner to another that is 10 blocks north, and 4 blocks east. How many possible routes are there to do this?

2. (**Triangular Graph**) In the triangular array,



   connect each dot by an edge to the two nearest dots just below it. At each dot, write the number of "edge-paths" downward to it from the top dot. What is the number in 15^{th} row, the 5^{th} dot from the left?

3. (**Walk on the line**) On the number line, starting at 0, you are to take14 steps, each of which is either distance 1 to the right, or distance 1 to the left, and in such a way that you end up at -6. How many ways are there to do this?

4. (**Unifix towers**) Using 10 white and 4 red unifix cubes, how many different 14-cube towers can you make?

5. (**Soccer score progressions**) The home team won a soccer game 10 to 4. What are all the possible sequences of scoring as the game progressed?

6. (**Choosing a team**) In a class of 14 students, you need to select a 4-student team. How many different ways are there to do this?

7. (**Balls in 2 bins**) What are all ways of putting 14 balls into two bins so that 10 balls are in bin A and 4 in bin B?

8. (**Cutting a ribbon**) You are to cut a 15-inch ribbon into five pieces, each of length a whole number of inches. How many ways are there to do this?

9. (**Binomial Theorem**) In the polynomial $(1 + y)^{14}$, what is the coefficient of y^4 ?

Expanded Usiskin CSPS (1/2 = 1/n + 1/m)

1. Find all ways to express ½ as the sum of two unit fractions (i.e. fractions of the form 1/n, n a positive integer).
2. Find all rectangles with integer side lengths whose area and perimeter are numerically equal.
3. Which pairs of positive integers have harmonic mean equal to 4?(*)
4. Nan can paint a house in n days, and Mom can paint it in m days (n and m positive integers). Working together they can paint the house in 2 days. What are the possible values of n and m?
5. The product of two integers is positive and twice their sum. What could these integers be?
6. Given a point P in the plane, find all n such that a small circular disk centered at P can by covered by non-overlapping congruent tiles shaped like regular n-gons that have P as a common vertex.
7. For which integers n > 2 does n – 2 divide 2n?
8. For which positive numbers s does $p(x) = x^2 - sx + 2s$ have integer roots?
9. Two vertical poles, A and B, have heights a meters and b meters, respectively, with a and b integers. A wire is stretched from the top of pole A to the base of pole B, and another wire is stretched from the top of pole B to the base of pole A. These wires cross at a point 2 meters above the ground. What are the possible values of (a, b)?
10. The base b and corresponding height h of a triangle are integers. A 2 x 2 square is inscribed in the triangle with one side on the given base, and vertices on the other two sides. What are the possible values of the pair (b, h)?
11. Let u be a positive real number. Find all solutions (n, m, v) with n and m positive integers, and v > 0, of the equation: $(uv)^2 = u^n = v^m$

(*) The harmonic mean h of n numbers $a_1, a_2, \ldots, a_n$: 1/h is the average of $1/a_1, 1/a_2, \ldots, 1/a_n$

Measure Exchange CSPS

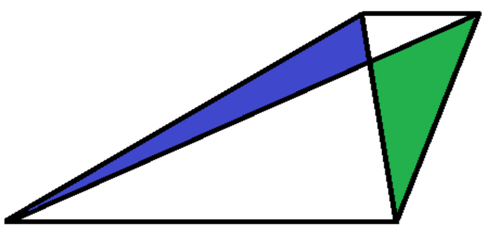
1. (**Tea & wine**) I have a barrel of wine, and you have a cup of green tea. I put a teaspoon of my wine into your cup of tea. Then you take a teaspoon of the mixture in your teacup, and put it back into my wine barrel. Question: Is there now more wine in the teacup than there is tea in the wine barrel, or is it the other way around?

2. (**Heads up**) I place on the table a collection of pennies. I invite you to randomly select a set of these coins, *as many as there were heads showing in the whole group*. Next I ask you to turn over each coin in the set that you have chosen. Then I tell you: The number of heads now showing in your group is the same as the number of heads in the complementary group. Question: How do I know this?

3. (**Faces up**) I blindfold you and then place in front of you a standard deck of 52 playing cards in a single stack. I have placed exactly 13 of the cards face up, wherever I like in the deck. Your challenge, *while still blindfolded*, is to arrange the cards into two stacks so that each stack has the same number of face-up cards.

4. (**Triangle medians**) In a triangle, the medians from two vertices form two triangles that meet only at the intersection of the medians. How are the areas of these two triangles related? More precisely, let ABC be a triangle. Let A' be the mid-point of AC, B' the mid-point of BC, and D the intersection of AB' and BA'. How are the areas of AA'D and BB'D related?

5. (**Trapezoid diagonals**) The diagonals a trapezoid divide the trapezoid into four relation of the areas of the two triangles containing the legs (non parallel sides) I



Conclusions and Questions

- Anecdotally: The students report that they find the course challenging, but worthwhile and engaging.
  - "*Your course made me think about math in ways I never had before.*"
  - *"I hope to be a better math teacher because of it."*
- I have shown you some of the curricular materials, but not the pedagogy, which is still experimental. It requires a lot of time, interaction, and scaffolding.
- Assessment: I do not yet have well designed measures of the effectiveness of this curriculum and instructional design. Suggestions welcome.

Thank you!

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The Euclidean Algorithm for finding gcd(a, b)

To find $d = \gcd(a, b)$ we can assume that $a \geq b \geq 0$.

Then put $a_0 = a$ and $a_1 = b$. If $a_1 = 0$ we have $d = a_0$.

If $a_1 > 0$ we apply Division w Remainder to write:

$$a_0 = q_1 a_1 + a_2, \quad \text{with } q_1 \text{ an integer} \quad \text{and} \quad 0 \leq a_2 < a_1.$$

Since $a_2 = a_0 - q_1 a_1$, it follows that

$$\gcd(a_0, a_1) = \gcd(a_1, a_2), \quad \text{with} \quad a_1 > a_2 \geq 0$$

If $a_2 = 0$ then we have $d = a_1$. If $a_2 > 0$ then we repeat the process to obtain

$$\gcd(a_1, a_2) = \gcd(a_2, a_3), \quad \text{with} \quad a_2 > a_3 \geq 0,$$

etc., finally producing:

$$\gcd(a_0, a_1) = \gcd(a_i, a_{i+1}) \quad (i = 1, 2, \ldots, n)$$

with

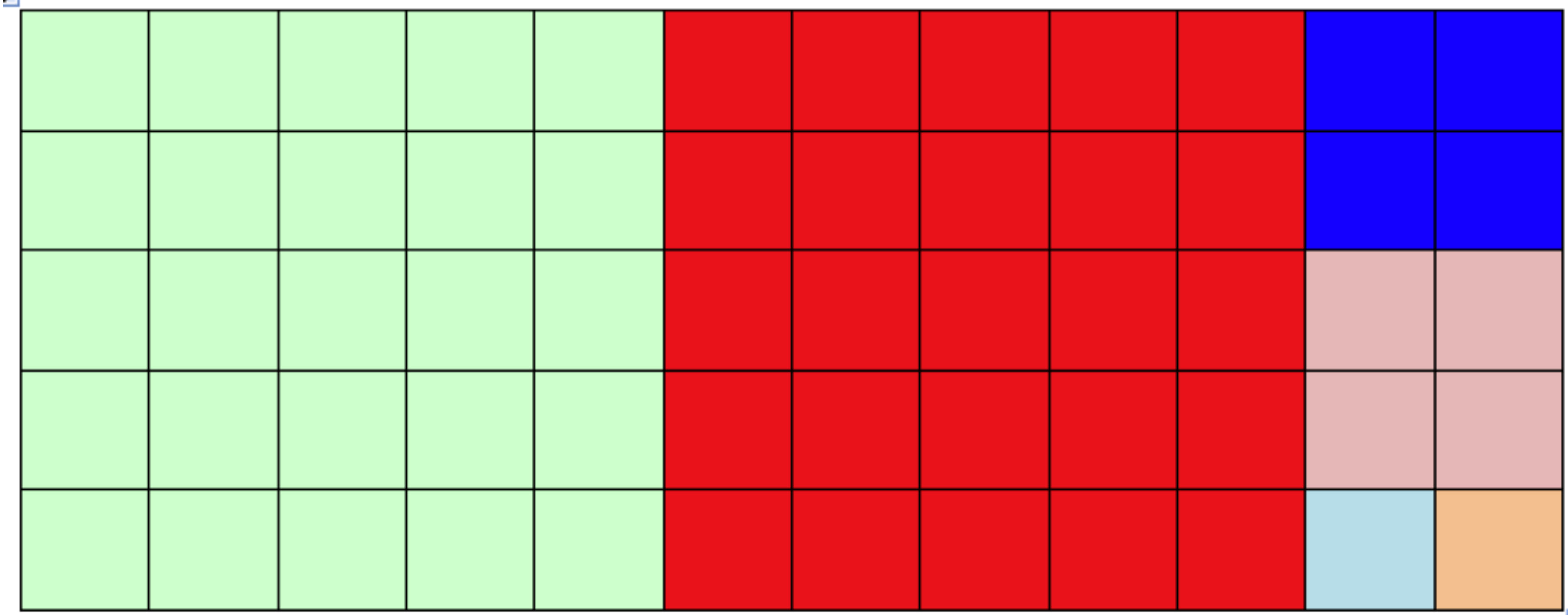
$$a_0 > a_1 > \ldots > a_n > a_{n+1} = 0, \quad \text{and} \quad d = a_n.$$

Euclidean Algorithm for gcd(a, b) <—> Square Tiling of an a x b Rectangle

"Greedy Algorithm" for Square Tiling

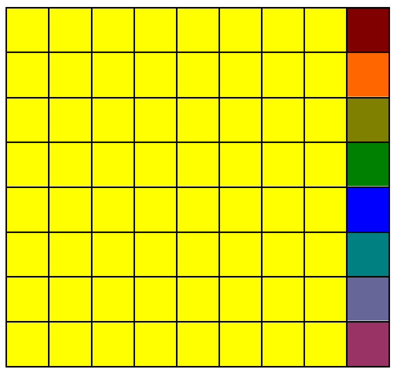
Keep cutting off the biggest possible squares

Euclidean Tiling of a 12 x 5 Rectangle



How many squares to tile a rectangle? (The 8 x 9 case)

The Greedy (Euclidean) tiling: 1 + 8 = 9 tiles



Fewer: 7 tiles