A Bachelor of Science in Mathematics that Emphasizes Mathematical Meanings for Teaching Mathematics

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RIME

Project Aspire: Defining and Assessing Mathematical Knowledge for Teaching Secondary Mathematics

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A Problem

- Students leave high school with poorly formed meanings for ideas of the secondary mathematics curriculum
- Students take mathematics courses that presume they have meanings they in fact do not have
- Students apply coping mechanisms in college math that allowed them to succeed in high school mathematics (memorization)
- Students return to high schools to teach ideas they understood poorly, rarely revisited, and for which they still have poorly-formed meanings

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Overview

- Illustrate problems with mathematical meanings drawn from a survey of 260 high school mathematics teachers
- Draw implications for undergraduate mathematics education and for teacher preparation
- Describe ASU's Bachelor of Science in Mathematics with math education concentration
  - Focus on Mathematical Meanings for Teaching secondary mathematics
  - Focus on ways to lead high school students' development of coherent mathematical meanings and ways of thinking

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Teachers' Mathematical Meanings

MMTsm
(Mathematical Meanings for Teaching secondary mathematics)

- A 43-item diagnostic instrument: Six animated items, nine multiple choice, 28 free-response items
- Covers ideas of function, function notation, functions as models, equation, quantitative reasoning, rate of change, proportionality, frames of reference, variation and covariation, and representational equivalence
- Administered to 160 high school teachers (summer 2012), 100 high school teachers (summer 2013). Teachers drawn from southwest and midwest U.S.
- Reliability and validity being established in conjunction with BEAR group at UC Berkeley.

Attending to Mathematical Meanings

A college science textbook contains this statement about a function *f* that gives a bacterial culture's mass at moments in time.

The change in the culture's mass over the time period Δx *is 4 grams.*

Part A. What does the word "over" mean in this context?

"During"

Part B. Express the textbook's statement in mathematical notation.

$$f(x+\Delta x)-f(x)=4$$

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Attending to Mathematical Meanings

The change in the culture's mass over the time period Δx is 4 grams.

Part A. What does the word "over" mean in this context?

divide.
Δx is the
bottom part of the ratio.

B.S. Math Ed

Part B. Express the textbook's statement in mathematical notation.

$$\frac{mass}{\Delta x} = 4$$

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Attending to Mathematical Meanings

The change in the culture's mass over the time period Δx is 4 grams.

Part A. What does the word "over" mean in this context?

B.S. Math

m is dependent on time (t)

Part B. Express the textbook's statement in mathematical notation.

$$\frac{m}{t}$$

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Attending to Mathematical Meanings

The change in the culture's mass over the time period Δx is 4 grams.

Part A. What does the word "over" mean in this context?

Level A0	The teacher wrote, "I don't know", the scorer cannot interpret the teacher's response, or the response did not address the question.
Level A1	The response conveys the idea that "over" means division
Level A2	• The teacher wrote both "divide" and "during" as meanings for "over", or • The teacher wrote "during" but described the time interval as something other than the stated interval, or • The teacher alluded to a passage of time, but without directly stating an interpretation of the word "over"
Level A3	The teacher wrote "during" or something equivalent

Part B. Express the textbook's statement in mathematical notation.

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Attending to Mathematical Meanings

The change in the culture's mass over the time period Δx is 4 grams.

Part B. Express the textbook's statement in mathematical notation.

Level B0	• The teacher wrote "I don't know" or reworded the original sentence • The scorer cannot interpret the teacher's response, or • The response contained a mathematical expression that is not described in Levels B1-B3.
Level B1	The teacher wrote a quotient showing the change in mass divided by the change in x is equal to 4, or some rearrangement of that statement.
Level B2	• The teacher wrote a true, but vague, statement such as $\Delta m = 4$ • The response contained a combination of the idea of $\Delta m = 4$ and the notation $\Delta m/\Delta x = 4$
Level B3	Teacher wrote something equivalent to $f(x + \Delta x) - f(x) = 4$

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Attending to Mathematical Meanings

The change in the culture's mass over the time period Δx is 4 grams.

Part A. What does the word "over" mean in this context?

Part B. Express the textbook's statement in mathematical notation.

All HS Teachers

	B0	B1	B2	B3	BX	total
A0	8	2	0	0	0	10
A1	15	12	0	0	0	27
A2	2	7	1	0	0	10
A3	29	16	3	3	1	52
AX	0	0	0	0	1	1
total	54	37	4	3	2	100

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Attending to Mathematical Meanings

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A2	2	7	1	0	0	10
A3	29	16	3	3	1	52
AX	0	0	0	0	1	1
total	54	37	4	3	2	100

54 of 62 teachers (87%) who said that "over" means "during" wrote nonsense or an expression involving division by Δ*x*.

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Attending to Mathematical Meanings

The change in the culture's mass over the time period Δx is 4 grams.

Part A. What does the word "over" mean in this context?

Part B. Express the textbook's statement in mathematical notation.

B.S. Math

	B0	B1	B2	B3	BX	total
A0	2	1	0	0	0	3
A1	3	3	0	0	0	6
A2	1	1	0	0	0	2
A3	5	6	2	0	0	13
AX	0	0	0	0	1	1
total	11	11	2	0	1	25

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Attending to Mathematical Meanings

The change in the culture's mass over the time period Δx is 4 grams.

Part A. What does the word "over" mean in this context?

Part B. Express the textbook's statement in mathematical notation.

B.S. Math Ed

	B0	B1	B2	B3	BX	total
A0	3	1	0	0	0	4
A1	5	5	0	0	0	10
A2	0	3	0	0	0	3
A3	12	6	0	2	0	20
AX	0	0	0	0	0	0
total	20	15	0	2	0	37

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Attending to Mathematical Meanings

The change in the culture's mass over the time period Δx is 4 grams.

Part A. What does the word "over" mean in this context?

Not Math or Math Ed

	B0	B1	B2	B3	BX	total
A0	2	0	0	0	0	2
A1	10	3	0	0	0	13
A2	2	3	1	0	0	6
A3	18	5	1	1	2	27
AX	0	0	0	0	0	0
total	32	11	2	1	2	48

Part B. Express the textbook's statement in mathematical notation.

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Why is "Over" Important?

- "Over" is not important
- But the orientation to computational interpretations that it reveals *is* important

Function Notation

The functions *f*, *g*, and *h* are defined below.

$$f(u) = u^2 - 1$$

$$g(s) = 1 + \frac{f(2s+1)}{2}$$

$$h(r) = g(r) - 1$$

What is $h(2)$? Show your work.

It seems there is not a major problem with teachers' meanings for function nottion. However …

	Math	MathEd	Other	total
Incorrect	4	4	11	19
Correct	20	28	17	65
No Ans	0	0	1	1
total	24	32	29	85

Function Notation (Summer 2012)

Here are two function definitions.

$$w(u) = \sin(u-1) \text{ if } u \geq 1$$

$$q(r) = \sqrt{r^2 - r^3} \text{ if } 0 \leq r < 1$$

Here is a third function c, defined in two parts, whose definition refers to w and q. Place the correct letter in each blank so that the function c is properly defined.

$$c(v) = \begin{cases} q(\underline{\ v\ }) \text{ if } 0 \leq \underline{\ v\ } < 1 \\ w(\underline{\ v\ }) \text{ if } \underline{\ v\ } \geq 1 \end{cases}$$

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Response	*Math*	*MathEd*	*Other*	*total*
R U	5	6	8	19
V	3	5	2	10
Mixed — V, R, and U	0	2	2	4
I don't know (or equivalent) — teacher's words	0	1	1	2
No answer – nothing written	0	2	0	2
total	8	16	13	37

Function Notation (Summer 2012)

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Response (Taught ≥ Precalculus)	*Math*	*MathEd*	*Other*	*total*
R U	2	2	0	4
V	0	1	2	3
I don’t know (or equivalent) — teacher’s words	0	1	1	2
Mixed — V, R, and U	0	1	0	1
total	2	5	3	10

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Consistent with thinking about function notation idiomatically. "$w(u)$" is the function's name.

Response	*Math*	*MathEd*	*Other*	*total*
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Part B

James, a student in an Algebra 2 class, defined a function *f* to model a situation involving the number of possible unique handshakes in a group of *n* people. He defined *f* as:

$$f(x) = \frac{n(n+1)}{2}$$

According to James' definition, what is *f*(9)?

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Function Notation (Summer 2013)

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$$f(x) = \frac{n(n+1)}{2}$$

According to James' definition, what is $f(9)$?

50%

	Math	MathEd	Other	total
Level 0	12	12	23	47
Level 1	3	4	2	9
Level 2	10	21	10	41
Level X	0	0	1	1
total	25	37	36	98

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Distinction Between Input and Argument (Summer 2012)

The function h is strictly increasing, and $h(b - 5) = 9$ for some number b.

Which of $(b, 9)$ or $(b - 5, 9)$ is on the graph of $y = h(x - 5)$? Explain.

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Distinction Between Input and Argument (Summer 2012)

The function h is strictly increasing, and $h(b - 5) = 9$ for some number b.

Which of $(b, 9)$ or $(b - 5, 9)$ is on the graph of $y = h(x - 5)$? Explain.

input

Distinction Between Input and Argument (Summer 2012)

The function h is strictly increasing, and $h(b - 5) = 9$ for some number b.

Which of $(b, 9)$ or $(b - 5, 9)$ is on the graph of $y = h(x - 5)$? Explain.

argument

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Distinction Between Input and Argument (Summer 2012)

The function h is strictly increasing, and $h(b – 5) = 9$ for some number b.

Which of $(b, 9)$ or $(b – 5, 9)$ is on the graph of $y = h(x – 5)$? Explain.

Response	*Math*	*MathEd*	*Other*	*total*
$(b, 9)$	5	7	4	16
$(b - 5, 9)$	2	7	4	13
Function name as multiplication	3	1	2	6
No answer -- nothing written	0	1	1	2
9 - 5 = 9	0	0	1	1
I don't know	0	1	1	2
total	10	17	14	41

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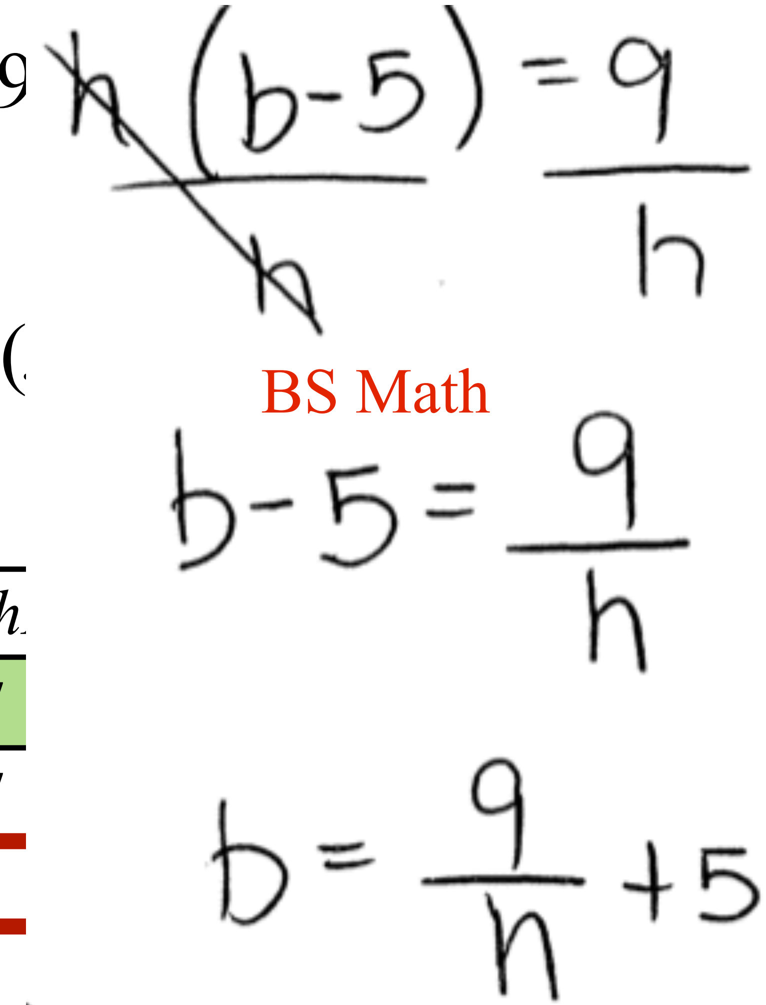
Distinction Between Input and Argument (Summer 2012)

The function h is strictly increasing, and $h(b - 5) = 9$
number b.

Which of $(b, 9)$ or $(b - 5, 9)$ is on the graph of $y = h($
Explain.



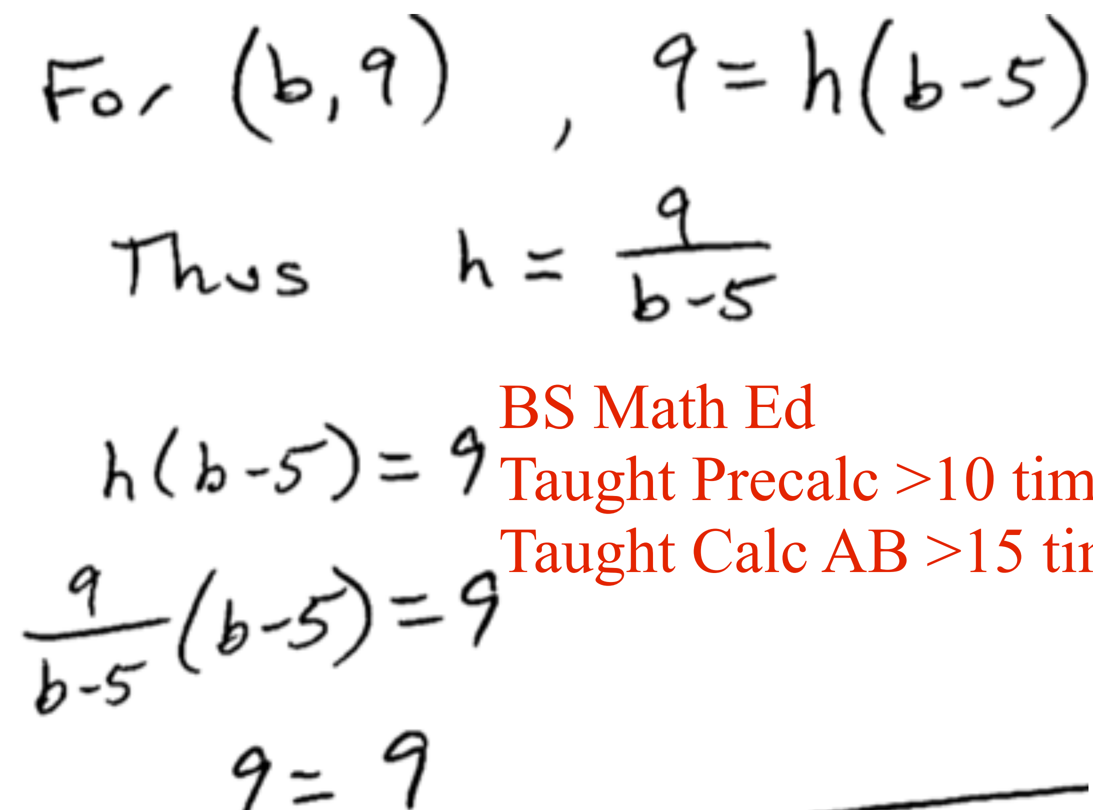
Response	*Math*	*Math*		
(b, 9)	5	7		
(b - 5, 9)	2	7		
Function name as multiplication	3	1		
No answer -- nothing written	0	1		
9 - 5 = 9	0	0	1	1
I don't know	0	1	1	2
total	10	17	14	41

Distinction Between Input and Argument (Summer 2012)

The function *h* is strictly increasing, and *h*(*b* – 5) = 9 for some number *b*.

Which of (*b*, 9) or (*b* – 5, 9) is on the g

Explain.



BS Math Ed
Taught Precalc >10 times
Taught Calc AB >15 times

Response				
(*b*, 9)	[illegible]	[illegible]	[illegible]	[illegible]
(*b* - 5, 9)	[illegible]	[illegible]	[illegible]	[illegible]
Function name as multiplication	[illegible]	[illegible]	[illegible]	[illegible]
No answer -- nothing written	0	1	1	2
9 - 5 = 9	0	0	1	1
I don’t know	0	1	1	2
total	10	17	14	41

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Distinction Between Input and Argument (Summer 2013, 87 HS Math Teachers)

The **function** h is strictly increasing, and $h(b - 5) = 9$ for some number b.

Which of $(b, 9)$ or $(b - 5, 9)$ is on the graph of $y = h(x - 5)$? Explain.

Results were Independent of Major

➜ 1. b is the input that gives 9 as the output so the point $(b, 9)$ is on the graph.
2. $(b, 9)$ is on the graph because the function is strictly increasing and the input to h is $b - 5$.
3. Solve for h in $h(b - 5) = 9$. When b is 6, then h is 9. So $(b, 9)$ is on the graph.
4. $b - 5$ is in the x position and 9 is in the y position so the (x, y) point is $(b - 5, 9)$.
5. $b - 5$ is the input that gives 9 as the output so the point $(b - 5, 9)$ is on the graph.

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Distinction Between Input and Argument (Summer 2013, 87 HS Math Teachers)

The **function** h is strictly increasing, and $h(b - 5) = 9$ for some number b.

Which of $(b, 9)$ or $(b - 5, 9)$ is on the graph of $y = h(x - 5)$? Explain.

Results were Independent of Major

	Response	%
➜	1. b is the input that gives 9 as the output so the point $(b, 9)$ is on the graph.	26%
	2. $(b, 9)$ is on the graph because the function is strictly increasing and the input to h is $b - 5$.	3%
	3. Solve for h in $h(b - 5) = 9$. When b is 6, then h is 9. So $(b, 9)$ is on the graph.	6%
	4. $b - 5$ is in the x position and 9 is in the y position so the (x, y) point is $(b - 5, 9)$.	7%
	5. $b - 5$ is the input that gives 9 as the output so the point $(b - 5, 9)$ is on the graph.	42%

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Interpreting Statements Quantitatively

Every second, Julie travels j meters on her bike and Stewart travels s meters by walking, where $j > s$. In *any* given amount of time, how will the distance covered by Julie compare with the distance covered by Stewart?

a. Julie will travel $j - s$ meters more than Stewart.

b. Julie will travel $j \cdot s$ meters more than Stewart.

c. Julie will travel j / s meters more than Stewart.

d. Julie will travel $j \cdot s$ times as many meters as Stewart.

e. Julie will travel j / s times as many meters as Stewart.

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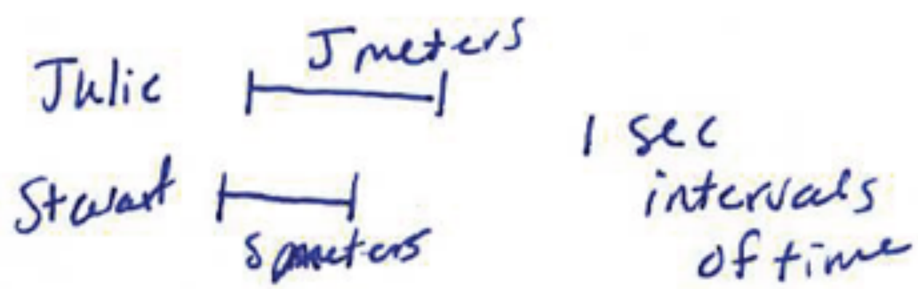

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ASU

Interpreting Statements Quantitatively

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e. Julie will travel j / s times as many meters as Stewart.

Response	Math	Math Ed	Other	Total
j-s	15	22	10	47
j*s more	0	0	4	4
j/s more	4	5	3	12
j*s times	0	1	5	6
j/s times	5	10	15	30
"no time"	1	0	0	1
Total	25	38	37	100

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Interpreting Statements Quantitatively

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j*s times	0	1	5	6
j/s times	5	10	15	30
"no time"	1	0	0	1
Total	25	38	37	100

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Meaning in Undergraduate Mathematics

- Assessment covered ideas of function, function notation, functions as models, rate of change, quantitative reasoning, equation, proportionality, frames of reference, variation, and representational equivalence.
- In all areas, a majority of teachers who took substantial undergraduate mathematics *from mathematics departments* showed weak meanings like those demonstrated today.
- It seems reasonable to conclude that these teachers had these weak meanings while enrolled in their undergraduate math courses.

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Meaning in Undergraduate Mathematics

- Assessment covered ideas of function, function notation, functions as models, rate of change, quantitative reasoning, equation, proportionality, frames of reference, variation, and representational equivalence.
- In all areas, a majority of teachers who took substantial undergraduate mathematics *from mathematics departments* showed weak meanings like those demonstrated today.
- It seems difficult to conclude that these teachers developed these meanings *after* leaving their undergraduate mathematics courses.

Meaning in Undergraduate Mathematics

- Assessment covered ideas of function, function notation, functions as models, rate of change, quantitative reasoning, equation, proportionality, frames of reference, variation, and representational equivalence.
- In all areas, a majority of teachers who took substantial undergraduate mathematics *from mathematics departments* showed weak meanings like those demonstrated today.
- It seems reasonable to conclude that these teachers had these weak meanings while enrolled in their undergraduate math courses.
- We must wonder what sense they made of higher mathematics when they had little conceptual foundation for it.

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Meaning in Undergraduate Mathematics

- Assessment covered ideas of function, function notation, functions as models, rate of change, quantitative reasoning, equation, proportionality, frames of reference, variation, and representational equivalence.

- In all areas, a majority of teachers who took substantial undergraduate mathematics *from mathematics departments* showed weak meanings like those demonstrated today.

We are now testing this

- It seems reasonable to conclude that these teachers had these weak meanings while enrolled in their undergraduate math courses.

- We must wonder what sense they made of higher mathematics when they had little conceptual foundation for it.

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ASU's Response

Bachelor of Sciences in Mathematics with concentration in Math Ed

- Bachelor of Science in Mathematics
- Graduates with this degree are licensed by AZ Dept of Ed to teach secondary mathematics
- Focused "like a laser" on future teachers' *mathematical meanings for teaching secondary mathematics* **and** on *students' mathematics*
- Specialized courses in mathematics education *in the math department*
- Faculty in Math Ed: Pat Thompson, Marilyn Carlson, Luis Saldanha, Kyeong Hah Roh, Carla Van de Sande, Fabio Milner, Mark Ashbrook, Stacy Musgrave

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Specialized Courses in Math Education (Mathematics Department)

- Algebra and Geometry in the High School (1st semester)
  - Mentored Tutoring
- Calculus developed according to Harel's *Necessity Principle* (1st semester)
- Technology and Mathematical Visualization (3rd semester)
- Mathematics Curriculum and Assessment in Grades 7-12 (5th semester)
- Development of Mathematical Thinking (7th semester)
- Methods of Teaching Secondary Mathematics (7th semester)

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Algebra and Geometry in the High School (1st semester)

- Central meanings in high school mathematics and how they can be built coherently (Marilyn Carlson's Pathways curriculum)
- Six hours of tutoring per week in ASU's precalculus tutoring center
- Coordination between course and tutoring

Start the process of building images of others' mathematics

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Technology and Mathematical Visualization (3rd semester)

- Creating didactic objects — artifacts that are designed to support reflective conversations about important mathematical ideas and ways of thinking
- Students must use mathematics to create them
- Focus simultaneously on future teachers' mathematics and their future students' mathematics
- Emphasize lesson design and how to hold classroom mathematical conversations
- Examples

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Curriculum & Assessment (5th semester)

- Teaching Gap (Stigler & Hiebert, 1999)
- International comparisons of curricula (Schmidt and others)
- Examination of other countries' elementary and secondary textbooks (Japan, Singapore, Finland)
    - Emphasize development of mathematical meanings and ways of thinking over time
- Learning goals and forms of assessment
    - Didactic triad
    - Formative and summative assessments; examples from US and other countries

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Development of Mathematical Thinking (7th semester)

- Research on additive and multiplicative reasoning, and their development
- Design and conduct a teaching experiment with one school student on a foundational mathematical idea
- Analyze and report findings

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Thank You

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