1. Effective equidistribution of certain adelic periods

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Let $\mathcal{B}_n$ be the space of positive definite forms on $\mathbb{R}^n$ up to $\sim$, where $B_1 \sim B_2$ iff $B_1 = \lambda g B_2$, $g \in GL_n(\mathbb{Z})$. We can identify it with $PGL_n(\mathbb{Z})\backslash PGL_n(\mathbb{R})/PO_n(\mathbb{R})$.

Given an integral form B, we call B' to be in the same genus of B iff they are locally equivalent, i.e. (1) $\forall$prime p, $\exists g \in GL_n(\mathbb{Z}_p)$ such that $B = gB'$, and (2) $\exists g \in GL_n(\mathbb{R})$ such that $B = gB'$.

Remark 1: (1) and (2) implies that $\exists g \in GL_n(\mathbb{Q})$ s.t. $B = gB'$. Hence, condition (2) can be replaced with: $\exists g \in GL_n(\mathbb{Q})$ such that $B = gB'$.

Remark 2: $B \sim B'$ then they have the same genus, but having the same genus does not imply $\sim$. For example, let $B = \begin{pmatrix} 5 & 0 \\ 0 & 11 \end{pmatrix}$, $B' = \begin{pmatrix} 1 & 0 \\ 0 & 55 \end{pmatrix}$, then $g = \begin{pmatrix} 1/4 & -1/4 \\ 1/4 & 5/4 \end{pmatrix}$ takes care of the infinite place as well as all finite places except for 2, which can be taken care of by $\begin{pmatrix} 1/7 & -22/7 \\ 2/7 & 5/7 \end{pmatrix}$, hence they are in the same genus. However, $B \not\sim B'$ because $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 11 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 55 \end{pmatrix}$ implies $5a^2 + 11b^2 = 1$ which has no integer solution.

From now on $n \geq 3$.

Remark 3: B and B' are in the same genus then they have the same determinant, hence each genus is a finite set in $\mathcal{B}_n$.

Theorem 1: Let $\{B_i\}$ be a sequence of integral forms mutually not locally equivalent, and their genus $\to \infty$, then the genus of B_i equidistributed in $\mathcal{B}_n$ with a rate of a power of $gen(B_i)^{-1}$.

Remark 4: No splitting condition is required in the statement of this theorem. If splitting at finite places is assumed the qualitative part of the result follows from earlier works.

Proof: This is based on an adelic equidistribution result.

Let F be a number field, $\mathbb{G}$ a semisimple connected algebraic group over F, $\mathbb{H}$ a semisimple connected, simply connected group. Fix map $\iota : \mathbb{H} \to \mathbb{G}$ which is a homomorphism with finite kernel over F, $\iota(\mathbb{H}) \subset \mathbb{G}$ is an algebraic subgroup.

Example: Consider $SL_2/\mathbb{Q}$ action on $sl(3)$. It is not an isomorphism from SL_2 to its image although the image is the same as SL_2 at all places.

Let $X = \mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A})$, $Y = \iota(\mathbb{H}(\mathbb{Q})\backslash\mathbb{H}(\mathbb{A}))g$, μ_x a $\mathbb{G}(\mathbb{A})$-invariant probability measure, and μ_Y the $g_i^{-1}\mathbb{H}(\mathbb{A})g = H_y$-invariant measure on Y. We want to understand what happens when Y gets complicated. There are two ways for Y to get "complicated": through a complicated $\mathbb{H}$ or a complicated g. In both cases the stabilizer is changing.

We can measure the complicity of Y as follows: let $\Omega_0 \subset \mathbb{G}(\mathbb{A})$ be a fixed bounded open set containing e, define $vol(Y) = m_Y(\Omega_0 \cap H_y)^{-1}$.

Theorem 2: $\exists\delta$ depending on $\dim_{\mathbb{F}} G$ and $[F : \mathbb{Q}]$ (only $\dim_{\mathbb{F}}(\mathbb{G})$ if assume the quotient is compact) s.t. (**) if $\iota(\mathbb{H}) \subset \mathbb{G}$ is maximum, $\forall f \in C_c^\infty(X)$, if $\mathbb{G}$ is simply connected, $|\mu_Y(f) - \mu_X(f)| < C_f vol(Y)^{-\delta}$, C_f is a constant depending only on f. Here, smooth means smooth on real places and invariant under a compact open subgroup.

Proof: $Stab(\mu_Y) = g^{-1}\iota(\mathbb{H}(\mathbb{A}))\mathbb{L}(F) = N_{\mathbb{G}}(\iota(\mathbb{H}))$. $\mathbb{L}$ can be infinite.

Assume $F = \mathbb{Q}$. For almost every prime p, $H_Y(\mathbb{Q}_p)$ is not compact. Let $U \subset H_Y(\mathbb{Q}_p)$ be a one-parameter unipontent subgroup. Y has large volume implies $\exists x, y \in Y$, $y = xg$, such that $|g|$ is bounded by some negative power of $vol(V)$, and $g \notin Stab(\mu_Y)$. Due to spectral gap, i.e. $\mathbb{H}(\mathbb{Q})$ action on $L^2(\mathbb{H}(\mathbb{Q})\backslash\mathbb{H}(\mathbb{A}))$ being $1/M$ tempered, most points on Y are effectively generic for U.

Now, by u_t action we get $g' \in \mathbb{G}$ such that $|\mu_Y(f) - \mu_Y^{g'}(f)|$ is bounded by a constant depending on f by a negative power of $vol(Y)$, In order to make $u_t^{-1} g u_t$ has size $O(1)$, we need to control p with regards to $vol(Y)$, which is accomplished by using the result of Prasad and Borel-Prasad: $1 = \tau(\mathbb{H}) = |\omega|(\mathbb{H}(\mathbb{Q})\backslash\mathbb{H}(\mathbb{A}))D_F^{-\dim\mathbb{H}/2}$. Prasad gives a product formula for ω and Borel-Prasad studied $\mathbb{L}$.