1. Geodesic Planes in hyperbolic 3 manifolds

This is based on joint work with McMullen and Muhammadi.

Question: $M = \Gamma\backslash\mathbb{H}^3$ is a complete hyperbolic 3-manifold, $f : \mathbb{H}^2 \to M$ is a totally geodesic immersion, what are the possible closures of its image. It is known in finite volume case due to Ratner, Shah.

Let Λ_Γ be the limit set of Γ, $Core(M) = \Gamma\backslash Hull(\Lambda_\Gamma)$ be the core of M, M is called rigid acylindrical if $Core(M)$ is a compact submanifold with totally geodesic boundary. The components of $M - Core(M)$ are called "ends". This condition implies that $\partial\mathbb{H}^3 - \Lambda_\Gamma$ is a union of countably many open discs, and Λ_Γ is a Serpinski curve. Here, "rigid" means that when double the core it is rigid, and acylindrical is in the sense defined by Thurston.

Theorem: If M is rigid acylindric, the closure of $f(\mathbb{H}^2)$ can be either: (1) $f(\mathbb{H}^2)$ or (2) M, or (3) the closure of an end.

In the first case, $f(\mathbb{H}^2)$ may be either (a) compact, (b) cocompact nonelementary, or (c) an $\mathbb{H}^2$ embedded in one end.

In particular, $\overline{f(\mathbb{H}^2)}$ is always a manifold (possibly with boundary).

Example: In the Fuchsian case, i.e. $M = S\times\mathbb{R}$, given any geodesic $\gamma \subset S$, the orthogonal plane of S containing γ is an immersed $\mathbb{H}^2$ hence its closure can be as complicated as the closure of geodesics on hyperbolic surface, which shows that rigid acylindrical is necessary. We can also bend it along a closed geodesic to make it quasi-Fuchsian and Γ Zariski dense.

To relate it with homogenuous dynamics, consider the frame bundle $F(M) = \Gamma\backslash PSL_2(\mathbb{C})$. Let $G = PSL_2(\mathbb{C})$ Then the set of totally geodesic hyperplanes without orientation is $G/PGL_2(\mathbb{R})$. Hence, we need to classify the $H = PGL_2(\mathbb{R})$-orbit closures in $\Gamma\backslash G$, or, equivalently, the Γ-orbits on the set $\mathcal{C}$ of circles on S^2.

Theorem (M,M,O) Let $C \subset S^2$ be a circle, then there are 5 cases:
(1) ((1)a above) $C \cap \Lambda = C$, in which case $\Gamma(C)$ is discrete and the stablizer of C in Γ is cocompact in H up to conjugation.
(2) ((1)b above) $C \cap \Lambda$ is a thin Cantor set. $\Gamma(C)$ is discrete and $Stab_\Gamma(C)$ is convex cocompact in H up to conjugation.
(3) ((2) above)$C \cap \Lambda$ is a thick Cantor set. $\overline{\Gamma(C)} = \{D \in \mathcal{C} | D \cap \Lambda \neq \emptyset\}$
(4) ((3) above)$C \cap \Lambda$ is one point, and C is inside a disc B which is a component of $S^2 - \Lambda$ except for that point. $\overline{\Gamma(C)} = \{D \in \mathcal{C} | D \cap \Lambda \neq \emptyset, D \subset \gamma\overline{B}\}$, where $\gamma \in \Gamma$.
(5) ((1)c above)$C \cap \Lambda = \emptyset$. In which case $\Gamma(C)$ is discrete.

Here, a Cantor set X is "thin" means $\exists k > 0$, such that any r-ball centered on $x \in X$ contains a kr-ball disjoint from X. If a Cantor set is not thin we call it thick.

This is related to the the unipotent flow on frame bundles. Let the set of renormalized frame bundle $RFM = \{x \in \Gamma \backslash G | \overline{xA} \text{compact}\}$, where A is the diagonal subgroup.

Theorem 2: $x \in RFM$, then xH is closed or $\overline{xH} = RFM \cdot NH$, where N is the upper triangular group.

The proof is related to Margulis' argument and based on the following recurrence lemma:

Lemma: Let $u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $\exists K$ such that $\forall y \in RFM$ $\{t | yu_t \in RFM\}$ is K-thick.

Here, set I is K-thick means $\forall x$, $([-Kx, -x] \cup [x, Kx]) \cap I \neq \emptyset$.

A further problem is the measure classification.