1. Local spectral gap

Joint work with Boutonnet and Ioana.

Let G be a compact Lie group, $\Omega \subset G$ a symmetric finite subset with counting measure μ_G, $\Gamma =< \Omega >$ is dense in G, $T_\Omega : L^2(G) \to L^2(G)$ is defined as $f \mapsto \mu_\omega * f$. T_Ω is self adjoint and has largest eigenvalue 1. The spectral gap is the gap between 1 and the next largest point in the spectrum. i.e., spectral gap implies that there are no "almost invariant" functions under the random walk. The breakthrough on spectral gap is due to the work of Boargain-Gambord.

Theorem (Benoist-de Saxce) G is a compact simple Lie group, $\Omega \subset GL(g)$ is algebraic, $< \Omega >$ is dense, then there is spectral gap.

Remark: "algebraic" implies certain Diophantine-type property.

Theorem (SG) (p-adic case) $\Omega \subset GL_{n_0}(\mathbb{Q})$, $\Gamma =< \Omega >$ is infinite and Zariski dense in $\mathbb{G}$, $\lambda(\mu_\Omega; GL_{n_0}(\mathbb{Z}_p)) < \lambda_0$ iff $\mathbb{G}^0 = [\mathbb{G}^0, \mathbb{G}^0]$ for all primes p where $\Gamma \subset GL_{n_0}(\mathbb{Z}_p)$.

To generalize it to locally compact case, need to fix a "window" and define the concept of local spectral gap.

Definition: Γ is a countable group with measure-preserving action on (X, μ), it has local spectral gap iff $\exists B \subset G$, bounded with non-empty interior, and $\exists \Omega \subset \Gamma$ finite and constant $C > 0$, such that $\forall f \in L_0^2(B)$, $\max_{\gamma\in\Omega} ||\gamma f - f||_{2,B} \geq c||f||_{2,B}$.

Theorem: spectral gap is equivalent to strong ergodicity.

Remark: When G is compact, the existence of spectral gap is equivalent to the existence of local spectral gap.

Theorem (B-I-SG) G is a simple Lie group, $Z(G) = 1$, Γ is algebraic in $GL(g)$, and is dense in G, then the Γ action on G has local spectral gap.

Application to orbit equivalent rigidity:

Definition: $\Gamma \subset G$ and $\Lambda \subset H$, where Γ and Λ are countable dense subgroups, are orbit equivalent if there is a measurable map sending Γ-orbits to Λ-orbits. If this map is an isomorphism we call them conjugate. If orbit equivalence implies conjugation for all $\Lambda \subset H$, we call $\Gamma \subset G$ orbit equivalence rigid.

Theorem: $\Gamma \subset G$ is dense, algebraic, G is a simple Lie group with trivial center, then $\gamma \subset G$ is orbit equivalence rigid.

Application on Banach-Ruziewicz problem:

Theorem: $\Gamma \subset G$ is a dense countable subgroup. The Γ-action has local spectral gap iff the Haar measure is the only Γ-invariant finite additive measure.

Theorem: (Restricted spectral gap) G, Γ, B as before, then $\forall \epsilon > 0$, $\exists \Omega \subset B_\epsilon(1) \cap \Gamma$ and a finite dimensional subspace $V \subset L^2(B)$ s.t. $T_\Omega : L^2(B) \to L^2(B)$ has norm $< 1/2$ on the orthogonal complement of V.