1. Genericity along curves and applications

Joint work with Fraczek, Shi. Two applications of homogenuous dynamics and flat surfaces in mathematical physics.

Application 1: Billiards in ellipses: For billiards in ellipses, every trajectory stays tangent to a "caustic" which is a cofocal conic determined by the starting point and angle. Let I_λ be the set of trajectories tangent to $\mathcal{C}_\lambda$. This is a genus 1 surface and the billiard dynamics is integrable. Dragovic and Radnovic studied billiards in ellipses with an additional barrier which is part of an axis, whose dynamics is "semi-integrable" and admits invariant sets I_λ which are higher genus surfaces.

Theorem (D-R) $\exists\lambda$ such that trajectories in I_λ is dense but not equidistributed.

Theorem (A1) (F-S-U) for a.e. λ, the billiard flow is uniquely ergodic on I_λ.

Application 2: Eaton lenses are circular "perfect retroreflectors". They are circular symmetric and sends any incoming light ray back in the opposite direction. Now consider placing Eaton lenses of radius R at lattices point of a lattice $\Lambda \subset \mathbb{R}^2$. We require them to be admissible i.e. the lenses do not intersect. A trajectory is trapped iff it stays in a bounded strip.

Theorem: fix $R > 0$, for a.e. admissible Λ all trajectories in the vertical direction is trapped.

Question: what if Λ is fixed?

Theorem (A2) Fixing any Λ, R admissible, the trajectory in almost every direction θ is trapped.

The trapped strip can be calculated by the eigenspaces of K-Z cocycle.

Both (A1) and (A2) follows from the Birkhoff and Oseledett genericity for curves in $ASL(2,\mathbb{R})/ASL(2,\mathbb{Z})$. $ASL(2,\mathbb{R}) = \begin{pmatrix} A & \xi \\ 0 & 1 \end{pmatrix}$ where $\xi = (\xi_1,\xi_2)^T$ and $A \in SL(2,\mathbb{R})$. $X = ASL(2,\mathbb{R})/ASL(2,\mathbb{Z})$ is the space of affine lattices. Consider geodesic flow $a_t = diag(e^t, e^{-t}, 1)$, it.is ergodic in X.

We call $x \in X$ is Birkhoff generic iff $\frac{1}{T}\int_0^T f(a_t x)dt = \int f d\mu$ for all $f \in C_c(x)$. Consider curves $\gamma(\lambda) = \begin{pmatrix} 1 & \lambda & \phi(\lambda) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Theorem (B1) (F-S-U) If $\forall$ rational line L, $\{\lambda | (\lambda, \phi(\lambda)) \in L\}$ has measure 0, then for a.e. λ, $\gamma(\lambda)$ is Birkhoff generic.

In (A1) by change of variable, it can be reduced to a rectangular billiard with a vertical barrier in the middle. However, when changing the direction of the trajectory of the original billiard, the direction of the trajectory of the new billiard is unchanged while its shape is changed. The translation surface corresponding to this billiard belongs to $\mathcal{M}_{dc} = \{$genus 2 surfaces that are double covers of a flat torus$\}$, which is a double cover of X, and a_t is the Teichmuller geodesic flow. Hence (A1) follows from (B1).

Theorem (B2) Ler $\tilde{\gamma}$ be a curve $[0,1] \to \mathcal{M}_{dc}$ which projects to a curve γ as in (B1), then for a.e. λ, $\tilde{\gamma}(\lambda)$ is Oseledett generic w.r.t. the K-Z cocycle.

(B2) is needed for (A2). There is a related result by Eskin-Chaka.

Idea behind the proof of (B1) and (B2): consider the week limit of measure, show that they're invariant under unipotent flow, then rule out the case where they have lower dimensional support by suitable height function.