1. Bernoulli convolutions for algebraic parameters

Joint work with Brevillard.

Fix $0 < \lambda < 1$, let $\xi_0, \xi_1, \dots$ be independent, unbiased ± 1 valued random variables, then the Bernoulli convolution μ_λ is the law of $\sum_{i=1}^{\infty} \xi_i \lambda^i$. This is the stationary measure of $x \mapsto \lambda x \pm 1$.

Question: is it absolutely continuous or singular?

$\lambda < 1/2$: supported in a Cantor set, singular. $\lambda = 1/2$: Lebesgue measure on $[-2, 2]$. $\lambda > 1/2$: not fully known.

Theorem (Erdos) If λ^{-1} is Pisot, μ_λ is singular. (proved by Fourier transform.)

Theorem (Solonyak) for a.e. $\lambda \in [1/2, 1]$, μ_λ is absolutely continuous.

Theorem (Hochman) $\{\lambda | 1/2 < \lambda < 1, \dim \mu_\lambda < 1\}$ has packing dimension 0.

Theorem (Shinerikin) $\{\lambda | 1/2 < \lambda < 1, \mu_\lambda \text{ is singular}\}$ has Hausdorff dimension 0.

$\mu_\lambda = \mu_{\lambda^k} * v$, for some v, hence μ_{λ^k} absolutely continuous, so is μ_λ. In particular, $\mu_{2^{-\frac{1}{k}}}$ are abs. continuous. Garcia gave other examples.

Theorem (Hochman) Suppose $1/2 < \lambda < 1$ is algebraic, $\dim \mu_\lambda = \min\{1, h_\lambda/(1 - \log \lambda)\}$, where $h_\lambda = \lim_{l \to \infty} \frac{1}{l} H(\sum_{i=0}^{l-1} \xi_i \lambda^i)$, H: Shannon entropy.

Theorem (B-V) λ algebraic, then $0.4 \min\{1, \log_2 M(\lambda)\} \leq h_\lambda \leq \min\{1, \log_2 M(\lambda)\}$. When λ is a unit (otherwise the entropy is always 1), $M(\lambda) = \prod_{\sigma : |\sigma(\lambda)| > 1} |\sigma(\lambda)|$, where σ goes through all algebraic embeddings.

Proof ideas: if X is absolutely continuous random variable on $\mathbb{R}^d$ with density f, then the Differential Entropy $H(x) = -\int f \log f dx$. Let $A \in GL_d(\mathbb{R})$, $H(X; A) = H(X + AG) - H(AG)$ where G is the standard Gaussian. $H(X; A_1 | A_2) = H(X; A_1) - H(X; A_2)$.

Proposition (1) $0 \leq H(X; A) \leq H(X)$. (2) $0 \leq H(X_1; A_1 | A_2) \leq H(X_1, X_2; A_1 | A_2)$ if $||A_1 x|| \leq ||A_2 x||, \forall x \in \mathbb{R}^d$, and X_1, X_2 are independent.

Let A be a matrix such that its eigenvalues are the Galois conjugates of λ. Let $X_{A,x}^{(l_1, l_2)} = \sum_{i=l_1}^{l_2} \xi_i A^i x$, then $H(X_{A,x}^{(0, l-1)}) = H(\sum_{i=1}^{l} \xi_i \lambda^i)$.

Lemma 1: $h_\lambda \geq \lim_{l \to \infty} \frac{1}{l} H(X_{A,x}^{(0,\infty)}; A^l | Id)$.

Lemma 2: $h_\lambda \geq H(X_{A,x}^{(0,\infty)}; A|Id)$. (Because $H(X_{A,x}^{(0,\infty)}; A^l|Id) = \sum_k H(X_{A,x}^{(0,\infty)}; A^k|A^{k-1})$, and the first term is the smallest.)

Lemma 3: $h_\lambda \geq H(\xi_0 x; A|Id)$. (by a matrix decomposition of A)

Lemma 4: $h_\lambda \geq H(\xi_0 t + G) - H(\xi_0 t + M(\lambda)^{-1} G)$. Now calculate and maximize the right hand side.