1. Metric diophantine approximation on Lie groups

Joint work with Roseizneig, de Saxce.

Definition: x is called β-diophantine if $\exists C$, and p, q arbitrary large, such that $|qx - p| < C/q^{\beta}$.

Diophantine exponent $\beta(x)$ is the inf of all β where x is β diophantine.

$\beta(x) = 1$ a.e. x and $\beta(x) = 1$ when x is algebraic (Roth).

For tuples, let $x \in \mathbb{R}^n$ and $q \in \mathbb{Z}^n$. Now $\beta \geq n$.

Question: What is the tuple satisfies some constrains, i.e. $x \in M \subset \mathbb{R}^n$, where M is a submanifold?

Example: Mahler's curve $(x, x^2, \dots x^n)$. The generic β for this is n.

Theorem (Kleinbock-Margulis) If M is analytic and not in a proper affine subspace, then $\beta(x) = n$ for a.e. x. We call such M "extremal".

Theorem (Kleinbock) If M is analytic $\exists \beta$ s.t. $\beta(x) = \beta$ a.e. x, and β only depends on the affine span of M.

Now let $x \in M_{m,n}$, then x is β-diophantine if $\exists C$, $\exists q \in \mathbb{Z}^n$ s.t. $||qx|| \geq C/||q||^{\beta}$. In this case, $\beta \geq n/m - 1$. In general, the generic β as above exists.

Obstruction to being extremal: Example: 4 vectors in $\mathbb{R}^2$, 3 of them in the same direction.

Definition: $W \subset \mathbb{R}^n$ is a subspace, $r \in \mathbb{Z}^+$, the pencil of endomorphism $P_{w,r} = \{y \in M_{m,n} | \dim(y(W)) \leq r\}$.
Remark: If W is rational, $\psi \in P_{W,r}$, then $\beta(y) \geq \dim(W)/r - 1$. (by the proof of Dirichlet's principle)

Define $r_{\mathbb{Q}} = \max\{dim(W)/r, W \text{ rational}, M \subset P_{W,r}\}$, $r_{\mathbb{R}} = \max\{dim(W)/r, M \subset P_{W,r}\}$.

Theorem 1: $r_{\mathbb{Q}} \leq \beta(M) + 1 \leq r_{\mathbb{R}}$.

Remark: this was proved in a more general setting.

Example: Let $u_1, \dots u_4$ be 4 vectors in $\mathbb{R}^3$, $y = (u_i \wedge u_j) \subset M_{6,3}$ is in a pencil yet still extremal.

(BKM) Not in amy pencil $\implies$ extremal.

Remark: If $rank(Y) = m \leq n$, $y \in P_{W,r} \equiv W \cap ker(Y)$ has dimension $\geq dim(W) - n \equiv ker(Y) \subset$ a Schubert variety. Also, β only depends on the kernel, so it depends only on its image in the Grassmannian. Because the image of Schubert varieties are affine in Plucker embedding to $\mathbb{P}(\wedge^d \mathbb{R}^n)$, we have:

Theorem 2: $\beta(M)$ depends only on the affine span of its image in $\mathbb{P}(\wedge^d \mathbb{R}^n)$. In particular, it depends only on the Zariski closure of M.

Conjecture: $\beta(M) = \beta(\cap$ pencils containing $M)$.

How to compute it?

$\phi : Grass_d(\mathbb{R}^n) \to \mathbb{Z}$: $w \mapsto \max_{Y \subset M} dim(Y(w))$ is non-decreasing and submodular $(\phi(w_1 + w_2) + \phi(w_1 \cap w_2) \leq \phi(w_1) + \phi(w_2))$.

Submodular lemma: $G \subset GL_n$, if ϕ is submodular and G-invariant then the maximum of $\phi(W)/dim(W)$ is obtained in a G-invariant subspace.

Apply it on $Gal(\mathbb{C}/\mathbb{Q})$, then if the Zariski closure of M is defined over $\mathbb{Q}$, then $\beta(M) = r_{\mathbb{R}} - 1 = r_{\mathbb{Q}} - 1$.

Motivation: Diophantine approximation on Nilpotent Lie groups. Pick a sequence of elements, how close can a word of them be close to identity?