1. Escape of mass for measures invariant under the diagonal group

Setting: $X_{n+1} = SL(n+1,\mathbb{R})/SL(n+1,\mathbb{Z})$ the space of lattices. Let $A = \exp(\mathbb{R}_0^{n+1})$ be the diagonal group. Thin part: $X^{<\delta} = \{x | \exists v \neq 0, v \in X, ||v|| < \delta\}$.

Theorem: $\exists$ sequence of lattices x_k, all with compact A-orbits, and $\mu_{Ax_k} \to 0$ weak*. Moreover, $\{x | \exists a_k \in A, x = \lim_{k\to\infty} a_k x_k\} \subset A\mathbb{Z}^{n+1}$, moreover, $\exists \delta_k \to 0$, s.t. $Ax_k \cap X^{\geq \delta_k}$ has at least $n!$ components.

This is very different from the unipotent case. Also, $n = 1$ is due to continuous fraction.

Question: Can there be partial escape of mass? Can the weak* limit be non-ergodic w.r.t. its invariant group?

How to show escape of mass? Take a compact orbit A_x, let $\Delta_x = Stab_A(x) \subset \mathbb{R}_0^{n+1}$ covers a large part of $\mathbb{R}_0^{n+1}$ with shape that will go to thin part.

Proposition: Let $\Delta = span_{\mathbb{Z}}\{t_l, l = 1, \dots n+1\} \subset \mathbb{R}_0^{n+1}$, $\sum t_l = 0$, S is the convex hull of t_l, then $\Delta + (n/2)S$ covers $\mathbb{R}_0^{n+1}$ and $n/2$ is sharp. $\mathbb{R}_0^{n+1} \backslash (\Delta + n/2(1-\rho)S)$, where $\rho > 0$, is contained in $\Delta + B_{c\rho r}(W_\tau)$, where r is the max of $||t_l||$, τ is a permutation, W_τ is τ applied on $\frac{1}{n+1}\sum_k k t_{k+1}$.

Criteria on lattice with full escape:

x_r is a family of lattices with compact A-orbit, if (1) the stablizer of x_r, Δ_{x_r}, is spanned by $t_1, \dots t_{n+1}$, $\sum t_l = 0$, and the max of the coordinates of t_l is less than $r + O(1)$, (2) $|\Delta_{x_r}| >> r^\epsilon$ for some $\epsilon > 0$, and (3) $\exists v_r \in x_r$, s.t. $||v_r|| << e^{-nr/2}$, then:
(1) $Ax_r = \{a(t)x_r\}$; (2) $\{a(t)x_r | t \in \frac{n}{2}(1-\rho)S_r\} \subset X^{\leq \delta}$; (3) $\mu_{Ax_r}(X^{\geq \delta}) << r^{-\epsilon/2}$, where $\delta = O(e^{-\frac{n}{2}r^{\frac{\epsilon}{2n}}}), \rho = O(r^{-1+\frac{\epsilon}{2n}})$.

How to find such x?

Given $\alpha > 0$, let Σ be the set of integer vectors in $\mathbb{Z}_0^{n+1}$. $\Sigma_\alpha = \{m \in \mathbb{Z}_0^{n+1} | m_i - m_j \geq \alpha ||m||\}$. Define $P_m(x) = \prod_l (x - m_l) - 1$.

Proposition: for all but finitely many m, P_m is irreducible, totally real, with roots $\theta_{m,l}$, such that $|\theta_{m,l} - m_l| = O(||m||^{-n})$.

Let x_m be $\begin{pmatrix} 1 & \theta_1 & \theta_1^2 & \dots & \theta_1^n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \theta_{n+1} & \theta_{n+1}^2 & \dots & \theta_{n+1}^n \end{pmatrix} \mathbb{Z}^{n+1}$ with covolume normalized.