## 1. Effective density of unipotent orbits

Joint work with Margulis, Muhammadi, Shah.

Theorem (conjectured by Raghunathan, proved by Ratner) $G$ Lie group, $\Gamma$ a lattics, $H \subset G$ closed subgroups generated by Ad-unipotent subgroups, then $\exists x \in G/\Gamma$ such that $\overline{Hg} = L[g]$ for some connected subgroup $L$.

$Lx$ is called periodic if $Stab_L x$ is discrete of finite covolume. Various special cases were proved by Dani, Margulis, Shah etc.

Similar result is true for algebraic group $G_S$ over $\mathbb{Q}_S$ where $S$ is a finite collection of places. This is proved by Ratner, Margulis, Tomanov.

Arithmetic case: $G$ linear algebraic group defined over $\mathbb{Q}$, no $\mathbb{Q}$-char, view it as embedded in the set of $n$-by-$n$ matrices. Let $S$ be a finite set of places, $\infty$ included, $\mathbb{G} = \prod_{v \in s} G_v$, $\Gamma \subset \mathbb{G}(\mathbb{Z}_S)$ of finite index, $H$ generated by 1 parameter unipotent group.

Theorem (Tomanov) $\overline{H[g]} = gL^0[e]$, $L^0$ fin. index of some $\mathbb{L}(\mathbb{Q}_S)$, for some $\mathbb{L}$ defined over $\mathbb{Q}$ generated by unipotents.

From Ratner's theorem, $f \in C_c(G/\Gamma)$, then $\lim_{T\to\infty} \int_0^T f(u_t[g])dt = \int_{L[g]} f$. Question: how fase does it converge in non-horospherical case? (horospherical case is known)

Example: $G = SL(2,\mathbb{R})$, $\Gamma = SL(2,\mathbb{Z})$. A big horocycle in a tube of periodic orbit$\to$ goes away$\to$ becomes equidistributed.

Let $\mathbb{G}$ linear algebraic group over $\mathbb{Q}$ semisimple, $\mathbb{G} = \mathbb{G}(\mathbb{R})^0$, $\Gamma$ arithematic lattice in $G$, $\alpha$ a height of $G/\Gamma$ induced by the map into $SL_N/SL_N(\mathbb{Z})$. $K_R = \{[g] \in G/\Gamma, \alpha(g) \leq R\}$, $\mathbb{H}$ defined over $\mathbb{Q}$, let $\mathcal{O}_{\mathbb{H}} \in \wedge^{dim\mathbb{H}}\}$, $hgt(\mathcal{H})$ is defined as $||\mathcal{O}||_\infty$, $\hat{\mathcal{O}} = \mathcal{O}/hgt$.

Theorem (MMSL): $u_t$ is a 1-parameter unipotent in a real algebraic normal subgroup $L < G$, $u_t$ has a dense orbit in $G/\Gamma$, $\forall T > R > 0$, $[g] \in K_R$, then either:
(1) $\exists \mathcal{H} < \mathcal{G}$ of class $\mathcal{H}$, not normal in $\mathcal{G}$, with height less than $R$ s.t. $\exists \gamma \in \Gamma$, $||u_t g \gamma \hat{\mathcal{O}_{\mathbb{H}}}|| < R, \forall |t| < T$. (i.e. orbit is trapped in a tube for a long time.)
(2) $\{u_t[g] : |t| < T\}$ is $1/\phi(g)$-dense, where $\phi = c \log\log\ldots\log t$, and there are $2(rkL + du(L))$ many log. $du$ is the dimension of the largest unipotent subgroup.

Proof is based on Dani-Margulis.